Solutions Manual

for

The Language of Mathematics

Nineteenth Edition

by Warren W. Esty

Solutions to the odd-numbered problems

Warren W. Esty

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10 9 8 7 6 5 4 3 2

Note: Many problems do not have a unique correct answer. The answer given here is among the best, but this is a language course, and languages permit different ways to say the same thing.

1) different letters may be employed to state the same solution. For example, some problems request the statement of theorems. Theorems stated using variables a, b, and c may be expressed using x, y and z.

2) different words may be used to express the same thought

3) pronunciations may vary

4) some problems request **new** examples, and your example might not (should not!) precisely duplicate the one in this manual

Chapter 1. Algebra is a Language

Section 1.1. Reexamining Mathematics

A1.	[Personal answer]				A3.	True
A5.	a) ex	b) s	c) ex	d) s		
A7.	a) s	b) ex	c) s	d) ex		
A9.	C $=2\pi r$	[or] $C = \pi d$			A11.	A = s^2 [or] A = x^2

A13. False

^^^^^^^^

B1. [Personal answer.] It is not really possible to acquire mastery of a language bit by bit. We cover everything again and again and anyone who sticks with it will get better and better.

B3. identity

B5. counterexample

B7. It expresses an assertion or fact.

B9. operations and order (and relations)

**** B11-14 [Your choice of example. This is just one possible example among many.]

B11. e.g. 9 - (-2) = 9 + 2 = 11 [Use different examples than these!]

B13. e.g. (2/3)(5/7) = 2(5)/(3(7)) =10/21

B15. 7/(3/5) = 7(5)/3 = 35/3

C1. Because it is used for communication and must be learned. It uses symbols to communicate about mathematical objects such as numbers, operations and order (functions), sets, set theory operations, etc.

Section 1.2. Order Matters!

A1. Order Matters!

A3. [Personal answer. Maybe PEMA]

A5. a) Yes. $(-7)^2 = 49$, but $-7^2 = -49$, since the negative sign comes last, because squaring is executed before multiplication, and the negative sign is regarded as multiplying by -1.
b) No, they are the same number. $(-1)^3 = -1$.

A7.	a) 5	b) -9	A9.	a) 14	b) 5
A11.	a) 9	b) -9	A13.	a) 4	b) -4
A15.	a) 8	b) 2	A17.	a) -1	b) 24

A19. a) 4.5 b) 1.5 c) 2.5 d) -4 e) 12
A21. a) 50 b) 27 c) -15
A23. a) c - (d/b) b) bc/d c) cd/b

**** A25ff. Minor variants are possible.
Some have many possibilities. This gives only one. To pronounce division, you may say "over" or "divided by". Some say “on”. “*a*/*b*” is then “*a* on *b*.”

A25. *x* is less than 3.
A27. three *x* [avoid "three times *x*"]
A29. *b c*
A31. *x* minus 2
A33. five *x* plus two
A35. *x* is less than or equal to 5
A37. negative 2 is less than *x* is less than 3
A39. *x* minus twelve equals negative 5
A41. three *x* squared
A43. negative [pause] three squared
A45. *a* plus [pause] *b* over 2
A47. 2 over the quantity *a* + *b*
A49. *a* minus the quantity *c* over *d*
A51. *b* over *c*, all over *d*
A53. *a* minus the quantity *b* minus *c*
A55. three over the quantity *x* plus 2
A57. *c d* squared
A59. *c* plus [pause, then more quickly] *d* squared
A61. two is less than *x* is less than 5
A63. one is less than or equal to x
A65. a b squared
A67. two x squared
Note: A69-72 are all hard to pronounce clearly
A69. the square root of the quantity *x* plus two [or] root [pause, then fast] *x* plus two
A71. the square root of the quantity two *x* plus 5
A73. sum A75. product A77. product A79. product

^^^^^^^^

B1. [Two requested] Horizontal fraction bar, $\frac{3+5}{2}$. Extended square root bar, $\sqrt{3+5}$
Superscripts, 3^2. Position and size make the difference.
B3. $ab = ba$. $a \times b = b \times a$ (for all *a* and *b*).
B5. 3b + 3c.
B7. $3(x + 9) = 60$ or $3x + 9 = 60$.
B9. no.
B11. $x - 20 + \pi$

B13. square, then multiply
B15. multiply, then add
B17. divide, then take the power
B19. add and then take square root

B21. F B23. T B25. T B27. F B29. F B31. F
B33. F B35. F B37. T

B39. F. $a^2 + 2ab + b^2$. B41. T
B43. F. No simplification. B45. F. $a - b + c$.

B47. 4 + 1 = × 3 = .
[There are other possible sequences on certain types of calculators, but this is a sequence which executes the operations in the functional order.]
B49. a) You may use: 4 + 3 × 5 squaring button =
since the calculator knows not to do the previous operations until all higher level operations have been done. The functional order is clearer in the sequence: 5 squaring button × 3 + 4 =
b) No!

^^^^^^^^

C1. Reciprocal before multiplication or addition.

Section 1.3. Reading Mathematics

A1. [Your answer is personal.]
A3. identity
A5. generalization
A7. unknown
A9. x equals 3
A11. 4 x equals 22. [do not say "4 times x ...]
A13. x minus 2 equals 15
A15. x over 3 equals 21 [or x divided by 3 equals 21
A17. five minus twelve equals negative seven
A19. five x equals 50
A21. three x minus two equals ten
A23. C equals pi d
A25. "=" between 15 and x should be "iff" or "$\Leftrightarrow$"
A27. "iff" should be "=".
A29. capital B.
A31. capitals should be lower-case.
A33. "=" between the two equations should be "iff"

A35. F A37. F A39. T A41. T A43. ex A45. s
A47. With a capital letter, Mathematics is the language (the symbolism) – the way of expressing thoughts. With a lower-case letter, mathematics is the content.

^^^^^^^^
B1. [See p. 8, p. 25, bold.] Unknowns represent numbers in sentences about numbers. Placeholders hold places in sentences about operations and order. e.g. $x + 5x = 12$ is about a number. $x + 5x = 6x$ is about operations and order.
B3. a) expressions (sequences of operations). b) equivalence of equations (sentences)
B5. a) [See p. 25] Two *expressions* with a variable are equivalent when they always express the same numerical value, regardless of the value of the variable.
b) For example, $(2x)^2$ and $4x^2$ are equivalent expressions.
c) The numbers are the same. The operations and order need not be the same.
B7. 1) as formulas: e.g. $A = \pi r^2$. 2) as identities: e.g. $-a - b = -(a + b)$.
B9. a) ex b) ex c) s d) s

B11. yes B13. yes B15. yes B17. no B19. no B21. yes

B23. U B25. P B27. U B29. U

^^^^ B31-39. Other letters could be used as placeholders. Each could have "For all ... " mentioned.
B31. $a + b = b + a$. B33. $-a - b = -(a + b)$.
B35. $1c = c$. B37. $b + 0 = b$.
B39. $x^2 \geq 0$. B41. It represents both a number and a sequence of operations.

^^^^^^^^
C1. Imperative sentences give a command. Declarative sentences make a true or false assertion. "Square both sides!" is imperative. "$a = b$ implies $a^2 = b^2$" is declarative.

Section 1.4. Algebra and Arithmetic

A1. $a+b = b+a$. commutative property (of addition)
A3. $ab = ba$. commutative property (of multiplication)
A5. $c(a+b) = ca+cb$. distributive property (of multiplication over addition)
A7. $(a + b) + c = a + (b + c)$, associate property (of addition)
A9. $(a + b) + c = a + (b + c)$, associate property (of addition)

A11. a) addition, multiplication. b) $a + b = b + a$, $ab = ba$ [Properties 2 and 7]
A13. a) subtraction, division. [Your examples may have particular numbers]
b) $3 - 1 \neq 1 - 3$. $4/2 \neq 2/4$. c) $a - (b - c) = a - b + c \neq (a - b) - c = a - b - c$, Property 31.
$a/(b/c) = ac/b \neq (a/b)/c = a/(bc)$, Properties 19.1, 19.3.
A15. a) $a(b + c) = ab + ac$. Distributive property. b) Property 9.
A17. a) $(a/b) + (c/d) = (ad + bc)/(bd)$. b) Property 18.
A19. a) Property 4. b) $12 = 3 + 9$.

A21. The "=" between "30" and "x" should be "iff".
A23. The "iff" should be "=".
A25. capital B
A27. middle "=" should be "iff"
A29. middle "=" should be "iff"
A31. iff" should be "=" A33. capital B and C

A35. x equals 5
A37. two x equals 50 [not “two times ....]
A39. x minus five equals eight
A41. x over 4 equals 20 [or] x divided by four equals 20
A43. three minus ten equals negative seven
A45. five x equals thrity
A47. two x minus five equals ten
A49. C equals pi d
A51. x squared equals c.
A53. x squared equals sixteen.
A55. T A57. not always true

A59. Only Property 13.

^^^^^^^^

B1. [See p. 26] a) Identities are about operations and order. The purpose of identities is to provide alternative sequences of operations for evaluating expressions. Identities express equivalence of expressions. (Equivalence of sequences of operations.)
b) The numbers are the same, but the sequence of operations is different.
B3. a) $x + 0 = x$. b) $ab = ba$

B5. $a - b = -(b - a)$ B7. $(-a)(b) = -(ab)$
B9. $a - (-b) = a + b$ B11. $(-a)(-b) = ab$
B13. $a/b + c/d = (ad + bc)/(bd)$ B15. $(a/b)/(c/d) = (ad)/(bc)$
B17. $(a/b)/c = \mathrm{a}/(bc)$ B19. $a/(-b) = -(a/b)$
B21. $a - (-b) = a + b$ B23. $a/b + c/d = (ad + bc)/(bd)$
B25. $(-a)(-b) = ab$ B27. $(a/b)/(c/d) = (ad)/(bc)$
B29. $(cx)/d$. Property 19.1. B31. $x - y + x$ (= $2x - y$). Property 31.
B33. b/h Property 19 B35. $x - a = b$ iff $x = b + a$.

**** B37-42 each have several possible answers. Also, different letters may be used to state the same answer.

B37. $x - a = b$ iff (is equivalent to) $x = b + a$.
[or] $a = b$ iff $a + c = b + c$.

B39. as B38. $x + a = b$ iff (is equivalent to) $x = b - a$.

B41. as B40. $x^2 = c$ iff $x = \sqrt{c}$ or $x = -\sqrt{c}$

^^^^ [Your example may not use the same numbers or letters as these.]

B43. $x^2 - 5 = 12$ sentence

B45. (3/5)/(7/11) expression

B47. a) Add the positive parts and attach a negative sign. b) $-a + (-b) = -(a + b)$.

B49. a) Invert and multiply. b) $(a/b)/(c/d) = (ad)/(bc)$.

B51. a) $a - b = a + -b$. b) Property 28.

B53. a) $a - b = x$ is equivalent to $x + b = a$. b) Property 4.

B55. Imperative sentences give a command. Declarative sentences make a true or false assertion. "Square both sides!" is imperative. "$a = b$ implies $a^2 = b^2$" is declarative.

B57. a) Add 34 to both sides.
b) $x - a = b$ iff (is equivalent to) $x = b + a$. [or] $a = b$ iff $a + c = b + c$.

B59. "the solution-pattern is exhibited in the other half."

^^^^^^^^

C1. a) $x^2 - y^2$ is incorrect. $x^2 - 2xy + y^2$ is right.
b) $1/(x + y)$ is incorrect [so is $2/(x + y)$]. $(y + x)/(xy)$ is correct.
c) $\sqrt{x} + \sqrt{y}$ is incorrect. There is no simplification.

C3. In a formula you are supposed to know what the letters mean. Their meaning is not given in the formula itself, although the letters used (for example, "A" for area) can help. On the other hand, a "relation between sentences" gives the original context as a problem-pattern. The solution pattern alone could be a formula.

C5. $a/x = c$ and $c \neq 0$ iff $x = a/c$ and $x \neq 0$.

C7. (Property 13.1) $x/b = c$ is equivalent to $x = bc$ and $b \neq 0$.

C9. 32+9 = (30+2)+9 [positional number system] = 30+(2+9) [Associative] = 30+11 [addition] = 30+(10+1) [positional number system] = (30+10)+1 [Associative] = (3(10)+1(10))+1 [multiplication – positional number system] =(3+1)10 + 1 [distributive] = 4(10)+1 [add] = 40+1 [mult] = 4 [positional number system]

Section 1.5. Reconsidering Numbers

A1. $x = -5$.	A3. $x = -5$.	A5. $x = -3$.	A7. $b = -7$.
A9. $x = -2$.	A11. $x = -2$.		
A13. T	A15. F, $x = -3$	A17. T	A19. F, $x = -2$
A21. F, $x = -10$	A23. F, $a = 0$, $b = -2$		

A25. the absolute value of x

A27. the absolute value of x [pause] minus 3 is less than 2

A29. the absolute value of x is less than c

A31. a) = b) iff c) $<$ d) iff e) $\Rightarrow$
A33. symbols don't point in the same direction
A35. capital B
A37. capital letters
A39. middle "=" sign
A41. symbols don't point in the same direction
A42. symbols don't point in the same direction
A43. "=" should be "iff"
A45. $c = 1$, $a = 3$, $b = -4$.

^^^^^^^^

B1. negative, zero, and fractions.
B3. a) location and directed distance.
b) "a" is a location, "b" is a directed distance, and "c" is a location.
"$a + b$" is b units to the right of a.
B5. a) The distance of x from the origin. b) The distance from b to c.
c) The distance from c to b. (= the distance from b to c)
d) The interiors are negatives of one another, so their absolute values are the same. It is just as far from b to c as it is from c to b.

B7. If a point is to the left of a second point, which is to the left of a third point, then the first point is to the left of the third.
B9. $x < 7/3$.
B11. $5 - 8 > x$. $-3 > x$
B13. $2x < 17+5 = 22$. $x < 11$.
B15. $-3x > 6$. $x < 6/(-3) = -2$. [or] $12 - 18 > 3x$. $-2 > x$.
B17. $|x - 5| < 3$ iff $-3 < x - 5 < 3$ iff $2 < x < 8$. $2 < x < 8$.
B19. $|2x - 1| < 3$ iff $-3 < 2x - 1 < 3$ iff $-2 < 2x < 4$ iff $-1 < x < 2$. $-1 < x < 2$.
B21. $|x| < 7$. $-7 < x < 7$.
B23. $|x - 3| < 7$. $-7 < x - 3 < 7$. $-4 < x < 10$.

B25. Yes. (True)
B27. Yes. (True)
B29. No. (False). $c = -55$ or $c = 15$.
B31. Yes. (True)
B33. False. For example, $-2 < 1$, but $|-2|$ is not $< |1|$.
B35. True.
B37. $c > 0$. B39. $a > 0$ B41. $0 < a$. B43. $c > 0$.

B45. $x = -5$. B47. $x = -3$. B49. $c = 0$, $x = 3$, $y = 4$. B51. $a = 0$, $b = 5$, $c = 8$.
B53. $k > 0$.
B55. $k \geq 1$ and $a \geq 0$ would suffice. [more precisely: ($k \geq 0$ and $a \geq -1$) or ($k \leq 0$ and $a \leq 1$)]
B57. $a \geq 0$. B58. $b > 0$. B59. $c \geq 0$. B61. $k \geq 0$.
B63. If $k \neq 0$. B65. If $c > 0$. B67. If $b > 0$. B69. If $c > 0$.

B71. a) 7. b) $2x - 4$.

B73. a) Divide both sides by 5. b) 1.5.3A c) 1/5, or 5 if read from left to right.
B75. a) Remove the abs value signs and get $-35.2 < x < 35.2$.
b) T1.5.5 c) 35.2
B77. a) Remove the abs value signs to get $-18.93 < x - 4.3 < 18.93$.
b) T1.5.5 c) 18.93 d) $x - 4.3$

B79. a) Divide both sides by 5.76.
b) T1.5.3A [better than T1.5.2A which does not use "iff"]
c) $c = 1/5.76$, $a = 5.76x$, $b = 27.1$
c) alternative: $c = 5.76$, $a = x$, $cb = 27.1$, so $b = 4.704...$ [b is not exhibited in the original problem.]

B81. a) Add 4.73 to both sides. b) x - b < c iff x < c + b.
b) has several alternatives, e.g. $a < b$ iff $a + c < b + c$.

B83. a) Subtract 1 from both sides. b) [T1.5.1B.] $x + c < b$ iff $x < b - c$.
c) x is $3x$, b is 1, c is 17. alternative for b): $a < b$ iff $a - c < b - c$.

B85. If $a > 0$, $ax < b$ iff $x < b/a$. If $a < 0$, $ax < b$ iff $x > b/a$.

B87. If b > 0 and c < d, then c/b < d/b.

B89. If $b \geq 0$ and $c \leq d$, then $bc \leq bd$.

B91. Whenever taking the absolute value is last on the lesser side of an inequality.

B93. This alternative has "$c \geq 0$", so at first it doesn't seem to address negative numbers.
The other one clearly addresses positive and negative numbers.
The other one uses "$-c$" for a positive number when $c < 0$, and that can be confusing.

^^^^^^^^

C1. Processes that work for solving equations do not necessarily work for solving inequalities. Addition and subtraction work the same in both cases, but multiplication and division do not. Whether or not the inequality remains in the same direction depends upon the sign of the multiplier. If the multiplier is negative, the direction must be changed. [This also means that multiplying through by an expression (such as "x") is illegal, since we do not know whether it is positive or negative. We can do that only by breaking it into separate cases, where $x \geq 0$ and where $x < 0$.]

C3. a) Let $c = -2$ and $x = 2$.
b) $|x| = c$ iff $c \geq 0$ and $x = c$ or $x = -c$.

C5. Yes. The direction of the inequality is reversed. The theorem: b < c iff d-b > d-c. The alternative theorem with "if...then...": "If b < c, then d-b > d-c," is not as useful because it does not assert equivalence.

C7. $|x - c| < d$ iff $-d < x - c < d$ by the Theorem on Absolute Values.
iff $c - d < x < c + d$, by T1.5.1.

C9. $c = (a + b)/2$ and $d = (b - a)/2$.
[From C7: $a = c - d$ and $b = c + d$. Solve these two equations for c and d.]

C11. $|x - 25| < 5$ C13. $|x - 6.6| < .1$

Chapter 2. Sets, Functions, and Algebra

Section 2.1. Sets

A1. a) $\{x|3 < x < 8\}$ b) $\{x|x \le 5\}$ c) $\{x|-2 \le x < 4\}$ d) $\{x|x \ge 9\}$
A3. a) $(-\infty,6)$ b) $[-3, -1.5)$ c) $[2.1,9.2]$ d) $[12,\infty)$ e) $[c,9)$
A5. The set of (all) x such that x is less than two.
A7. The set of (all) x such that x is greater than or equal to 3.
A9. x is in S. [or] x is a member of S.
A11. R is a subset of S.
A13. x is in the set of (all) x such that x is less than or equal to three.

A15. a) $(-7,12]$ b) $(2.1, 3]$ c) $(0,5.4)$
A17. a) $(3, 13]$ b) $(3.2, 4]$ c) $(2, 6.8)$

A19. a) $(-\infty, 3]$ b) $(-\infty, -2)$
A21. a) $(-\infty,0]\cup[3,\infty)$ b) $(-\infty,2)\cup[4.1,\infty)$

A23. a) T b) T c) F
A25. a) F b) F c) F

A27. $x \in S\cap T$. $x \in S$ and $x \in T$.
A29. $S \subset T$. If $x \in S$, then $x \in T$.

A31. a) Yes. (True) b) No. (False) (for example, $x = -50$); c) Yes. (True)
A33. a) False b) False c) False

A35. capital letters do not represent elements ($a \in S$ or $A \subset S$ would be ok)
A37. numbers are not subsets ($\{7\} \subset S$ or $7 \in S$ would be ok)
A39.] should be)
A41. "and" connects sentences, not sets.
A43. "$\cap$" connects sets, not sentences.
A45. "$x \in S$" is ok, but "$\Rightarrow$" cannot connect to a set.
A47. "$\cup$" connects sets, not sentences.
A49. fine
A51. "or" connects sentences, not sets, and $(8,\infty)$ is a set.
A53. members of T are represented by lower-case letters ($s \in T$ and $S \subset T$ would be ok)
A55. sets cannot equal numbers or sentences.
A57. members of S are represented by lower-case letters ($a \in S$ and $A \subset S$ would be ok)
A59. "$\Rightarrow$" does not connect sets, and "$S \cup T$" is a set.
A61. "$\Rightarrow$" does not connect sets, and "T" is a set.
A63. fine
A65. ∞ has), not]

^^^^^^^^

B1. a) [Figure 3] of *intersection* b) [Figure 4] of *union* c) intersection
B3. Sketch something like Figure 5 of *intersection*, *union*, and *complementation* (Figure 14 is also relevant).

B5. a) T = {2,5} has only two elements, 2 and 5. The other two are intervals of real numbers. S = [2,5] includes the endpoints. R = (2,5) does not.
b) R ⊂ S, T ⊂ S.

B7. and, intersection | or, union | not, complement

B9. [Sketch your own picture] with $a < b < c < d$. [Do not bother to label 0.]
a) $S \cap T = [b, c)$ b) $S \cup T = (a, d]$
c) $S^c = (-\infty, a] \cup [c, \infty)$ d) $R \cup T = [b, \infty)$
e) $T \subset S \cup R$

B11. a) $x \in S \cap T$. b) $x \in S$ and $x \in T$. c) iff

B13. a) $|x| < c$; $-c < x < c$.
b) they are true or false for the same values of the letters, so either can replace the other without changing the meaning.

B15. a) "7" is a number. "{7}" is a set with one member, the number 7.
b) $7 \in \{7\}$. $7 \not\subset \{7\}$.

B17. $(-\infty,3)\cup(3,7)\cup(7,\infty)$

B19. a) T b) F, $b = -6$ c) T d) F, $b = -6$

B21. a) F b) F c) T d) T

B23. $(-\infty,3) \cup (3,\infty)$.

B25. a) It should read $S \cap T = \{7\}$. Intersections form sets, not numbers.
b) It should read $T \subset S$. The members of S are numbers, not sets.

B27. -5

B29. It does not have one; 3 is not in it.

B31ff. [xx Insert Venn diagrams]

^^^^^^^^

C1. $A\text{-}B = A \cap B^c$ [Students may invent any notation for this concept, providing the notation does not have some other well-known meaning already. In some sense this concept is like subtraction, so notation resembling the minus sign of subtraction is commonly seen. The notation should incorporate "A" and "B".]

C3. "If" says which are in, "only if" which are not.

C5. A set is determined by its members, according to the definition of set equality. Listing a member twice would not change which members are in it, so there is no point in listing any member twice.

C7. We study concepts which are useful and fruitful.

Section 2.2. Functions

[**Note**: Letters may vary. When a problem asks "What notation would a mathematician use to define a function ..." the answer is given here using the letter "f " to denote the function, but other letters such as "g" or "h" are equally acceptable.]

A1. argument, function, image (in that order)

A3. argument - function - image

A5. a) Divide by 5! b) Add 4!

A7. a) Add 1 and then divide by 2! b) Add 2 and then square!

A9. a) Square it! b) Multiply by 4!

A11. a) $f(x) = |x - 1|$. b) $f(x) = x^2 + 1$.

A13. Note: Technically, the domain is a set, so set notation is appropriate. Nevertheless, these answers describe the domain.
a) all real numbers not equal to 2. b) all real numbers
c) all real numbers ≥ 5

A15. a) integers ≥ 2; b) all integers; c) all non-zero integers (n is for integers)

A17. a) $f(x) = x/2$. b) $f(x) = (x\text{-}5)/3$. c) $f(x) = 1/x$.

A19. a) $f(x) = (x/3)+4$. b) $f(x) = (x+4)/3$. c) $f(x) = 3x - 5$. d) $f(x) = 3(x\text{-}5)$.

A21. a) the area of a triangle is given by $A = (1/2)$bh.
b) The volume of a cylinder is given by $V = \pi r^2 h$.

A23. a) Multiply the lengths of the two sides. b) $A = bh$ [or lw].

A25. a) $f(x) = x^3$. b) $f(x) = 4x$.

A27. a, c, and d are for multiplication. b and e are for functions (f and g are typical function names).

A29. a) function b) integer c) number (image) d) number

A31. a) set b) number c) number d) number (image)

A33. f of x A35. f of x equals x squared A37. f of g of x

^^^^^^^^

B1. [See p. 90 or p. 98] for the definition of *function*.

B3. a) $f(x) + h$ applies f first and then adds h. $f(x+h)$ adds h first and then applies f. Order matters! b) Let $f(x) = 2x + 3$. Then $f(x) + h = 2x + 3 + h$, whereas $f(x + h) = 2(x + h) + 3 = 2x + 3 + 2h$.

B5. a) the argument and the function b) the argument and the image
c) functional notation

B7. They are the same number, but not the same sequence of operations.

B9. a) 16. b) z^2. c) $(x + 1)^2$. d) $(x/2)^2$ [$= x^2/4$].

B11. a) 3 b) $(x - 1)(x - 3)$. c) $x^2(x^2 - 2)$. d) $(x(x - 2))^2$ e) order matters!

B13. a) $2/8 = 1/4$ b) $2/(2/8) = 8$ c) $2/(2/z) = z$

B15. $f(x) = 20x$. B17. $f(x) = 5x$.

B19. $g(f(x)) = (\sqrt{x}) + 4$; domain all $x \geq 0$.

B21. $g(x) = \sqrt{x}$, $f(x) = x + 3$.

B23. $g(x) = 5x$, $f(x) = x - 7$.

B25. $h(x) = x^2$. $g(x) = \pi x$. $g(h(x)) = \pi x^2$.

B27. $h(x) = x + 2$. $g(x) = x^2$. $g(h(x)) = (x + 2)^2$.

B29. a) $V = s^3$ (volume of a cube of side s) b) $P = 4s$ (perimeter of a square of side s)

B31. $[3(x+h)+7 - (3x+7)]/h = 3h/h = 3$ if $h \neq 0$.

B33. b, if $h \neq 0$. $[b(x+h)+c -(bx+c)]/h = b$, if $h \neq 0$.

B35. $c(n) = 6n$ if $n \leq 10$; $c(n) = 60 + 5(n - 10)$ if $11 \leq n$. [$c(n) = 5n + 10$ if $11 \leq n$.]

^^^^^^^^

C1. log

C3. They are processes of mathematics. They indicate how to solve equations (p. 95). They are the objects of thought in identities. Also, when we do word problems (Section 2.4) we will see that, when numbers are unknown, we can still deal with the relevant functions (mathematical relationships) to build formulas that allow us to answer the problem using algebra.

C5. False. $-5 < 0$ but $f(-5) = 25$ is not $< f(0) = 0$.

C7. $f(x) = 2x + 1$ (there are others). C9. $f(x) = 2^x$. C11. $f(x) = x^3$.

Section 2.3. Solving Equations

A1. a, b, c, d, e
A3. a, d
A5. b, c, d, e [note: $x^2 + 1$ is never 0]

A7. N A9. Y A11. Y

A13. $3x + 5 = 38$ iff $3x = 33$ iff $x = 11$.
A15. $x^2 - 2 = 16$ iff $x^2 = 18$ iff $x = \pm\sqrt{18}$.
A17. $(x + 4)/5 = 3$ iff $x + 4 = 15$ iff $x = 11$.
A19. $3x = 20 - 2x$ iff $5x = 20$ iff $x = 4$.
A21. $x^2 = 55 - 2x^2$ iff $3x^2 = 55$ iff $x^2 = 55/3$ iff $x = \pm\sqrt{(55/3)}$.
A23. $x - 2 = 0$ or $x - 7 = 0$. $x = 2$ or $x = 7$.
A25. $2x - 6 = 0$ $x - 10 = 0$. $2x = 6$ or $x = 10$. $x = 3$ or $x = 10$.

A27. "a" is "$x - 5$", "b" is "12", and "c" is "5".
A29. "a" is "$(x - 3)^2 - 9$", "b" is "52", and "c" is "9".
A31. "a" is "$x^2 + 4$", "b" is "19", and "c" is "-4".

A33. $x^2 - 4x = -3$ iff $x^2 - 4x + 3 = 0$ Rule 2, Uniqueness of Addition
iff $(x-3)(x-1) = 0$ substitution = Rule 1
iff $x-3 = 0$ or $x-1 = 0$ Zero Product Rule = Rule 5
iff $x = 3$ or $x = 1$ Uniqueness of Addition, twice
A35. $x^2 = x + 20$ iff $x^2 - x - 20 = 0$ Uniqueness of Addition = Rule 2
iff $(x - 5)(x + 4) = 0$ substitution
iff $x - 5 = 0$ or $x + 4 = 0$ Zero Product Rule = Rule 5
iff $x = 5$ or $x = -4$ UA, twice
A37. $x(x - 5) = 2x$ iff $x - 5 = 2$ or $x = 0$ Canceling
iff $x = 7$ or $x = 0$ UA
A39. $(x^2 + 6x)(x-3) = 2x(x^2 + 6x)$ iff $x-3 = 2x$ or $x^2 + 6x = 0$ canceling
iff $-3 = x$ (subtract x from both sides, UA) or $x(x + 6) = 0$ sub
iff $x = -3$ or $x = 0$ or $x + 6 = 0$ ZPR = Rule 5
iff $x = -3$ or $x = 0$ or $x = -6$. UA

A41. From Rule 6 you can see right away that the solution must be $x = 9$ or $x = -9$. If so, that's great. But, for purposes of this problem, we want you to stick to Rules 1-5.
$x^2 = 81$ iff $x^2 - 81 = 0$ UA
iff $(x - 9)(x + 9) = 0$ sub
iff $x - 9 = 0$ or $x + 9 = 0$ ZPR = Rule 5
iff $x = 9$ or $x = -9$ UA, twice

A43. $x^2 + 3x = 0$ A45. $2x + 1 = x + 5$ or $x + 5 = 0$.
A46. $2x + 3 = 0$ A48. $x^2 + x - 6 = 6x$.
A49. $2x + 7 = x + 5$ or $x^2 + 3x - 42 = 0$.
A51. $x^2 - x - 20 = x + 6$. [or] $(x + 4)(x - 5) - x - 6 = 0$.

A55. "=" in the middle should be "iff".

A57. fine
A59. fine
A61. "=" in the middle should be "iff".

A63. "=" usually connects expressions, not equations.
A65. expressions cannot equal equations.
A67. expressions cannot equal equations.
A69. expressions cannot be equivalent to equations.
A71. "iff" connects sentences and {8} is a set.
A73. "$\Rightarrow$" connects sentences and {8} is a set.
A75. T A77. False (for negative x)
A79. 5/3 cannot solve Example 30 because the right side would be negative, and square roots are never negative.
A81. [See the definition on p. 107] A "corollary" of a result is another result that follows immediately, or almost immediately, from the original result.

^^^^^^^^

B1. a) The solutions to E1 are a subset of the solutions to E2. b) Yes. c) E2.
B3. a) It is preferable to have them connected by "iff".
b) Then the solution set is preserved.
c) Check the solutions to the terminal equation in the original equation and eliminate those which do not satisfy it. [That is, use the theorem on extraneous solutions to eliminate extraneous solutions.]
d) Theorem 2.3.7 on extraneous solutions.

B5. An algebraic identity is a special type of equation with a variable that is true for all values of the variable. An equation need not be true for all values of the variable, and it need not even have a variable.
B7. If the multiplying expression can take on the value 0, the new equation will be satisfied for the x which makes the multiplying expression 0, whether or not the x solves the original equation. The direction of Rule 3 warns us of this.

B9. $E \Rightarrow x^2 - 8x = 9$ squaring
iff $x^2 - 8x - 9 = 0$ iff $(x - 9)(x + 1) = 0$ substitution
iff $x - 9 = 0$ or $x + 1 = 0$ Zero Product Rule = Rule 5
iff $x = 9$ or $x = -1$ Addition, twice
Checking, both work.

B11. $x+1 = \sqrt{(13-2x)} \Rightarrow x^2+2x+1 = 13-2x$ Rule on squaring
iff $x^2+4x-12 = 0$ UA
iff $(x+6)(x-2) = 0$ sub
iff $x+6 = 0$ or $x-2 = 0$ Zero Product Rule
iff $x = -6$ or $x = 2$ UA, twice
Checking, $x = 2$ is the solution, since $x = -6$ is ruled out using T2.3.7.

B13. $\sqrt{(x + 3)} = x - 3 \Rightarrow x + 3 = (x - 3)^2$ squaring
iff $x + 3 = x^2 - 6x + 9$ sub
iff $0 = x^2 - 7x + 6 = 0$ UA
iff $0 = (x - 6)(x - 1)$ sub
iff $x - 6 = 0$ or $x - 1 = 0$ ZPR
iff $x = 6$ or $x = 1$ UA, twice
Checking, $x = 6$ is the solution. $x = 1$ fails to work; it is extraneous.

B15. $\sqrt{x} = 6 - x \Rightarrow x = (6 - x)^2$ squaring
iff $x = 36 - 12x + x^2$ sub
iff $0 = x^2 - 13x + 36$ UA
iff $0 = (x - 9)(x - 4)$ sub
iff $x - 9 = 0$ or $x - 4 = 0$ ZPR
iff $x = 9$ or $x = 4$. UA, twice
The solution is $x = 4$. $x = 9$ does not check.

B17. $\sqrt{(2x + 5)} = x - 5 \Rightarrow 2x + 5 = (x - 5)^2$ squaring
iff $2x + 5 = x^2 - 10x + 25$ sub
iff $0 = x^2 - 12x + 20$ UA
iff $0 = (x - 10)(x - 2)$ sub
iff $x - 10 = 0$ or $x - 2 = 0$ ZPR
iff $x = 10$ or $x = 2$.
The solution is $x = 10$. $x = 2$ does not check.

B19. $3x + \sqrt{(2x - 3)} = 5$ iff $\sqrt{(2x - 3)} = 5 - 3x$ (UA. This type of step must come before squaring, or else the square root will not disappear because there will be a cross-product term)
$\Rightarrow 2x - 3 = (5 - 3x)^2$ squaring
iff $2x - 3 = 25 - 30x + 9x^2$ sub
iff $0 = 28 - 32x + 9x^2$ UA
iff $(9x - 14)(x - 2) = 0$ sub (factoring)
iff $9x - 14 = 0$ or $x - 2 = 0$ ZPR = Rule 5
iff $9x = 14$ or $x = 2$ UA, twice
iff $x = 14/9$ or $x = 2$ Rule 4
Now, using the theorem on extraneous solutions, we see that $x = 2$ does not solve the original equation. With more work we would see that $x = 14/9$ does. It is the only solution.

B21. $x - 2 = \sqrt{(6x + 4)} \Rightarrow x^2 - 4x + 4 = 6x + 4$ squaring and substitution
iff $x^2 - 10x = 0$ UA
iff $x(x - 10) = 0$ sub
iff $x = 0$ or $x - 10 = 0$ ZPR = Rule 5
iff $x = 0$ or $x = 10$ UA
Checking, $x = 0$ does not work but $x = 10$ does. The solution is $x = 10$.

B23. $(x^2 - 6x + 8)/(x - 2) = 2x \Rightarrow x^2 - 6x + 8 = 2x(x - 2)$ UM
iff $x^2 - 6x + 8 = 2x^2 - 4x$ sub
iff $0 = x^2 + 2x - 8$ UA
iff $0 = (x + 4)(x - 2)$ sub
iff $x + 4 = 0$ or $x - 2 = 0$ ZPR
iff $x = -4$ or $x = 2$ UA, twice
Checking, $x = 2$ does not work. The solution is $x = -4$.
[Using the Rule on Eliminating a Quotient is very similar, it merely carries along an explicit statement that $x \neq 2$, which we discovered by checking]
[or] $(x - 2)(x - 4)/(x - 2) = 2x$, $x - 4 = 2x$ and $x \neq 2$, $x = -4$.

B25. $E \Rightarrow x^2 + 3x = 4$ UM
iff $x^2 + 3x - 4 = 0$ UA
iff $(x - 1)(x + 4) = 0$ sub
iff $x - 1 = 0$ or $x + 4 = 0$ ZPR
iff $x = 1$ or $x = -4$ UA, twice
Checking, $x = 1$ is not in the domain, so the solution is $x = -4$.
[Using the Rule on Eliminating a Quotient is very similar, it merely carries along an explicit

statement that $x \neq 1$, which we discovered by checking]
[or] ... iff $x^2 + 3x = 4$ and $x - 1 \neq 0$, and continue as before.

B27.	$E \Rightarrow -x^2 + 9 = -2x(x - 3)$	UM	
	iff $-x^2 + 9 = -2x^2 + 6x$	sub	
	iff $x^2 - 6x + 9 = 0$	UA	
	iff $(x - 3)^2 = 0$	sub	
	iff $x - 3 = 0$ or $x - 3 = 0$	ZPR	
	iff $x = 3$	UA.	Checking, there are no solutions.

[or, the left side could be factored and reduced, noting $x \neq 3$.]

B29.	$E \Rightarrow x^2 = 6x - 5$	squaring
	iff $x^2 - 6x + 5 = 0$	UA
	iff $(x - 5)(x - 1) = 0$	sub
	iff $x - 5 = 0$ or $x - 1 = 0$	ZPR
	iff $x = 5$ or $x = 1$.	UA, twice
	Checking, both work	

B31.	yes, substitution, iff	B33.	no
B35.	yes, UM, $\Rightarrow$	B37.	no
B39.	no	B41.	yes, UA, iff
B43.	no	B45.	yes, UA, iff
B47.	no	B49.	yes, substitution, iff
B51.	yes, squaring, $\Rightarrow$	B53.	yes, UM, $\Rightarrow$

B55. a) $E1 \Rightarrow E2$ b) $E2$
B57. $f(x) = h(x)$ or $g(x) = 0$.
B59. $h(x) = f(x)g(x)$ and $f(x) \neq 0$ (eliminating a quotient),
or $h(x) = f(x)g(x)$ (and remembering to check, by UM)
B61. $f(x) = g(x)$ or $f(x) = 0$.
B63. He or she has probably dropped a solution.
B65. a) any new example, e.g.: $x(x+9) = 3x$; $x+9 = 3$; $x = 6$, which dropped $x = 0$.
b) most likely Rule 8 on canceling

B67. $a^2 = ab$ iff $a = b$ or $a = 0$.

B69. If $c \neq 0$, $cx = d$ iff $x = d/c$.
B71. "-3 = 0" can be equivalent to another equation, in which case it is not a mistake. It is just <u>false</u>. That merely means that any equation equivalent to it is also (always) false, that is, has no solutions.
B73. Answer repeats answer to B72.
B75. Canceling like that is illegal. $x = 0$ is a solution, by Rule 8.

B77. B40 is just like B39, in that "2 = 2" is true for all x.

B79. $E2$ should read $x + 12 = x^2$ or $x - 5 = 0$ (the solution $x = 5$ was dropped)
"inspection" is not a rule. Another solution was dropped in going from $E2$ to $E3$ ($x = -3$).
B81. Let "c" be replaced by "$-c$" and simplify "$a + -c$" to "$a - c$."
B83. Let "c" be replaced by "$1/c$" (which we can do since $c \neq 0$), and then the right side becomes "$(1/c)a = (1/c)b$" which can be rewritten "$a/c = b/c$" by 1.4.13.

^^^^^^^^

C1. You might ask "Why use two *different* letters "a" and "b" when they are *equal*? Why not just use one letter, "$a = a$ iff $a + c = a + c$"? First of all, this rule does *not* assert that a and b are equal. It applies perfectly well if they are not equal. It says something *if* they are equal and it something else if they are not. But $a = a$ (so that version would assert they are equal). This is important. That means our version can be applied to any equation, even before you know the value(s) of "x" which make the equation true. Use of a single letter might suggest they should be identical, but they can be equal for some values of "x" and not equal for others. Thirdly, even when two expressions are equal, that fact may not be immediately evident. The two sides may appear different because they are expressed differently. [For example, $x^2 = 5x + 7$ has two sides which appear different, but we want to find the x's when they are equal.] Because Rule 2 is stated using two different letters it obviously applies to expressions which appear different, whether or not they are actually equal.

C3. Theorem (Rule): $(x - a)(x - b) = 0$ iff $x = a$ or $x = b$.
$(x - a)(x - b) = 0$ iff $x - a = 0$ or $x - b = 0$ ZPR
iff $x = a$ or $x = b$, UA, twice.

C5. Theorem: $x^2 = \mathrm{b}^2$ iff $x = \mathrm{b}$ or $x = -\mathrm{b}$.
$x^2 = \mathrm{b}^2$ iff $x^2 - \mathrm{b}^2 = 0$ iff $(x-\mathrm{b})(x+\mathrm{b}) = 0$ iff $x = \mathrm{b}$ or $x = -\mathrm{b}$

C7. If $a \neq 0$, $ax + b = c$ iff $x = (c - b)/a$.
Proof. If $a \neq 0$, $ax + b = c$ iff $ax = c - b$ (Rule 2) iff $x = (c - b)/a$ (Rule 4).

C9. $a/b = 0$ iff $a = 0$ and $b \neq 0$.
Proof: $a/b = 0 \Rightarrow a = 0$ [UM] Also, $a/b = 0 \Rightarrow b \neq 0$ [domain].
For the other half, if $b \neq 0$, $1/b$ exists and $a = 0 \Rightarrow a(1/b) = 0(1/b)$ [by UM], so, simplifying, $a/b = 0$.

C11. The proof has two halves. The Axiom itself is one. For the other:
$a + c = b + c \Rightarrow a + c + (-c) = b + c + (-c)$, which simplifies to $a = b$.

C13. See Example 3.2.17, and 3.3.15.

C15. $x = 7 \Rightarrow 0 = 0$ (subtracting itself), but the solution set to "0 = 0" is all x, so all the information of the original equation has been lost.

C17. a) If $a \geq 0$, $\sqrt{a} = c$ iff $c \geq 0$ and $c^2 = a$.
[If $a \geq 0$, there exists a real number $r \geq 0$ such that $r^2 = a$. Call that number $\sqrt{a}$.
i.e. If $a \geq 0$, $\sqrt{a}$ is that real number ≥ 0 such that $(\sqrt{a})^2 = a$.]
b) The domains of definition of the two sides are different.
[negative numbers are permitted on the right, but not on the left]

Section 2.4. Word Problems

A1. [See the definition on p. 129] a) Cue words are words that suggest a particular mathematical operation such as addition.
b) [See the definition on p. 130] A problem is direct when the given words, symbols, or basic formulas suggest the operations you actually do to solve a problem.

A3. [p. 133] Build your own formula.

A5. $\pi r^2/2 = 120$. $d = 2r$. $\pi(d/2)^2/2 = 120$. $\pi d^2/8 = 120$. [$d = 17.5$]

A7. Shorter side x. Other side 2x. A = bh = x(2x) = 50. $x^2 = 25$. $x = 5$.

A9. T = taxes. I = income. T = (1/10)(I - 14000). Also, T = 1200.
Solving, I = \$26,000. [You can also do this with x = "income above 14,000".]

A11. $x/16 = (x - 12{,}000)/10$. $10x = 16(x - 12{,}000) = 16x - 192{,}000$. $192{,}000 = 6x$. $x = 32{,}000$.

^^^^^^^^

B1. Name the unknown, possibly "x". Build your own formula.

B3. A cue word signals a relationship between quantities and is an operation to represent in a formula, but it is often not an operation to do.

B5. Let the first box be x tall. Second box is $2x$ tall. The third is $2x + 2$.
The stack is $x + 2x + 2x+2 = 42$. $5x = 40$. $x = 8$.

B7. a) Perimeter $= x + 2x + 12$. b) $x + 2x + 12 = 33$. [$x = 7$]

B9. In general, $P = 2x + 2y$, where x and y are the sides. a) Here, $P = 2x + 8$.
b) $20 = 2x + 8$. [$x = 6$.]

B11. a) $7 + 2(7) + \sqrt{(7^2 + 14^2)}$ [$= 36.7$]
b) $P = x + 2x + \sqrt{(x^2 + (2x)^2)}$. $x + 2x + \sqrt{(x^2 + (2x)^2)} = 50$. $x(1 + 2 + \sqrt{5}) = 50$.
[$x = 9.55$]

B13. a) $x^2 + 10^2 = 14^2$. $P = 10 + 14 + x = 24 + \sqrt{(14^2 - 10^2)}$.
b) $P = 10 + x + \sqrt{(x^2 - 10^2)} = 28$. c) b is indirect. [$x = 11.78$]

B15. a) $A = (1/2)bh$. $78 = (1/2)8h$. Solve for h. $h = 19.5$. Since $P = b + h + c$, we need c, where $c = \sqrt{(b^2 + h^2)}$. Then plug in for h, and 8 for b, to get P. [$h = 19.5$, $c = 21.08$, $P = 48.6$]
b) $A = (1/2)bh = 25$. $P = b + h + c = b + h + \sqrt{(b^2 + h^2)} = 26$.
use the first to solve for b or h, say h: $h = 50/b$. Plug in to the second:
$b + 50/b + \sqrt{(b^2 + (50/b)^2)} = 26$. This is solved by guess-and-check. [$b = 5.08$ or 9.84]

B17. Let x be his income. $.05x = .09(x - 8000)$. [$720 = .04x$. $x = 18{,}000$]

B19. a) $6(7) + 3(6)(1/2) = 51$.
b) Let the marked side be x. Drop a perpendicular to the base to form a rectangle of sides 6 and x and a triangle of sides $10\text{-}x$ and 6. Area $= 6x + 6(10\text{-}x)/2$.
$6x + 6(10\text{-}x)/2 = 50$. [$6x + 30 - 3x = 50$. $3x = 20$. $x = 6.67$.]

B21. Let the vertical side in Figure 7 be x. $A = xw$, where $4x + 2w = 100$.
$w = (100 - 4x)/2$. $A = x(100 - 4x)/2$ is the formula. You would maximize this.
[312.5 at $x = 12.5$.]

B23. $x + 2x + \sqrt{(x^2 + (2x)^2)} = 20$. [$x(1 + 2 + \sqrt{5}) = 20$. $x = 3.82$]

^^^^^^^^

C1. $A = x^2 + \pi(x/2)^2/2 = 10{,}000$. [$x = 84.73$.]

C3. Suppose you buy x tons. $50(.146) + .16x = .15(50 + x)$. [$x = 20$]
[or] [$50(.146) + .16x]/(50 + x) = .15$

C5. $200(.10) = 20$ (cubic yards of rocks in the original topsoil).
Buy x (cubic yards with 4% rocks).
Then the total amount of rocks will be $20 + .04x$ and the total topsoil will be $200 + x$. So $(20 + .04x)/(200 + x) = .06$ is the equation. [$x = 400$]
[or] $(20 + .04x) = .06(200 + x)$.

C7. $V = x^2h$. $4x + h = 150$, so $h = 150 - 4x$. Therefore $V = x^2(150 - 4x)$.
You would maximize this by guess and check. [max $= 31{,}250$ at $x = 25$]

C9. Surface area of picture: $24(16) = 384$. Half that $= 192$.
Total: $384 + 192 = 576$. Let the width be x.
$576 = (24 + 2x)(16 + 2x)$ [$x = 2.17$.]

C11. a) $A = (1/2)4h$. Bisect the base to form two right triangles, each with base 2 ($= 4/2$).
By the Pythagorean Theorem, $2^2 + h^2 = 6^2$. So $A = (1/2)4h = (1/2)4\sqrt{(6^2 - 2^2)}$ [$= 11.3$].
b) Let the unknown base be x. $A = (1/2)x\sqrt{(6^2 - (x/2)^2)}$. Set this $= 8$.
You would solve it with guess-and-check. [$x = 2.74$]

Chapter 3. Logic for Mathematics

Section 3.1. Connectives

A1. A sentence is a complete thought. An expression is a noun or a pronoun.
A3. b, c, d (f is an open sentence, but not a statement)

A5. a) expression b) expression c) statement d) expression
A7. a) statement b) expression c) expression d) expression

A9. b, c A11. b A13. b
A15. or

A17. H implies C (or, "If H, then C.")
A19. A and B implies C.

A21. $A \Rightarrow (B$ or $C)$ A23. (A and B) $\Rightarrow C$. A25. (not C) and A
A27. [H and (not C)] $\Rightarrow D$ A28. (not B) $\Rightarrow$ (not C) A29. $B \Rightarrow (A$ or $C)$
A31. $A \Rightarrow (C$ and $D)$

A33. If $x > 7$, then $x > 5$. A35. If B, then A. A37. If C, then D.
A39. If $x \in$ T, then $x \in$ S. A41. If $x > 9$, then $x \in$ S. A43. If C, then B.
A45. If C, then B.

A47. No, it connects sentences.

A49. "or" connects sentences, not sets, and $(8, \infty)$ is a set.
A51. "not" cannot apply to "x"
A53. ok, if A and C are sentences
A55. "or" connects sentences, not sets. (S is used to represent a set)
A57. "$\Rightarrow$" can not connect a number (as conclusion)
A59. fine
A61. "=" does not relate sentences and sets
A63. In this text (but not always in other texts) lower-case letters are members
($a \in S$ or $A \subset S$ would be ok)
A65. fine
A67. fine
A69. fine
A71. "x" is not a sentence
A73. B should be b
A75. should be lower case
A77. "$S \cap T$" is not a sentence
A79. fine
A81. "and" connects sentences, not sets.

**** The answers here are the most likely, but, in some contexts, these letters might refer to other types of mathematical objects. For example, it is possible for capital letters to represent numbers, although we have avoided that usage.
A83. a) sentence (possibly a set) b) set (possibly a sentence) c) function d) number
A85. a) interval (or, it could be an ordered pair) b) number
c) sentence (possibly a set) d) set

A87. a) expression (if a set) (could be a sentence in logic) b) open sentence
c) expression (set) d) connective

A89. a) sentence b) expression (set) c) sentence d) expression

^^^^^^^^

B1. a) Figures 2 and “and” and 3 for “or”. b) See Figure 4, with TT being row 1, etc.
c) "and"

B3. [Your choice. See Figures 6, 7, 5.]

B5. 4 B7. 6

B9. Your table should have 8 columns.

H	C	$C \Rightarrow H$	not C	not H	(not C)$\Rightarrow$(not H)	(not H)$\Rightarrow$(not C)	(not H) or C
T	T	T	F	F	T	T	T
T	F	T	T	F	F	T	F
F	T	F	F	T	T	F	T
F	F	T	T	T	T	T	T

Column 6 shows part b is LE to $H \Rightarrow C$. Columns 3 and 7 show a and c are LE. Column 8 prove Theorem 3.1.12.

B11. True. A two-row truth table proves it.

B13. a) T b) T c) T d) F

B15 & B16 & B17. Table for B15, B16 and B17

A	B	$A \Rightarrow B$	not B	not A	(not B)$\Rightarrow$(not A)	not($A \Rightarrow B$)	$A \Rightarrow$(not B)	$A \wedge$(not B)
T	T	T	F	F	T	F	F	F
T	F	F	T	F	F	T	T	T
F	T	T	F	T	T	F	T	F
F	F	T	T	T	T	F	T	F

Columns 3 and 6 show B15 is true.
Columns 6 and 9 show B17 is true.

B19. False. [Use a truth table.]

B21. 8 rows.

B23. a) complementation
b) intersection (also complementation, in a less important role)
c) union

^^^^ B25-B30. All could be misinterpreted. Order matters! All need parentheses, or at least a convention.

B25. T, T, F (A or B) $\Rightarrow C$

B27. A is F, B is F (not A) and B

B29. F, T, T no convention, use parentheses

B31. Put F's wherever C is F and F's wherever D is F. Put a T in the remaining rows.

B33. a) [A table will show it false.] Consider F, F, F.
b) No, "$\Rightarrow$ " is not associative.

B35. $x \le 5$ or $x^2 > 25$. B37. $x \notin S$ or $x \le b$.

B39. $(S^c)^c = S$.

B41. a) No, the associative law for "and" can be proved with an eight-line truth table.
b) Yes, "and" is associative.

^^^^^^^^

C1. a) A statement is true or false, but not both. Letters can represent particular statements. Then a combination of letters and connectives represents a compound statement. However, if letters are used as placeholders (and do not represent any statement in particular), then a combination of letters and connectives is a form (statement formula). b) "Is it true?"

C3. 4. A. not A. A and not A. A or not A.

C5. $2^8 = 256$. [8 lines in the table, each could be T or F.]

Section 3.2. Logical Equivalences

A1. [Your choice.] e.g. $x > 50 \Rightarrow x > 10$. A3. T A5. If $x^2 \le 100$, then $x \le 10$.

A7. "If not($x \le 12$), then $x \notin$ S," which is equivalent to "If $x > 12$, then $x \notin$ S."

A9. If it does not have an airbag, it is not a new Ford.
[or] If the Ford does not have an airbag, it is not new.

A11. a) and b) are converses; c) and d) are converses; a) and c) are contrapositives; b) and d) are contrapositives

A13. If $x \in$ S, then $x \in$ T.

A15. If $x \in$ R, then $x \le 36$.

A17. True.

A19. Yes. (That is half of the ZPR.)

^^^^^^^^

B1. By inspection of the truth table for "$H \Rightarrow C$," we see that it is true whenever H is false, so put a "T" in those rows. It is true whenever C is true, so put a "T" in those rows. Put a "F" elsewhere.
[Alternative: Note that $H \Rightarrow C$ is LE to (not H) or C. "Or" statements are true if either is true, so we can put a "T" in any row where "not H" is true and in any row where "C" is true. In the remaining row(s) it is false.

B3. If the complement of a set is not open, the set is not closed.

B5. If $f(x) \ge f(z)$, then $x \ge z$.

B7. If two angles are vertical, then they are congruent.
If two angles are not congruent, they are not vertical.

B9. If the student is a math major, the student is smart.
If the student is not smart, the student is not a math major.

B11. If the car is new, it has an airbag.
If the car does not have an airbag, it is not new.

B13. If $c > b > 0$, then $|c| > |b|$.

B15. If $x > 9$ and $f(x) = 3x + 5$, then $f(x) > 32$.

B17. If you attend MSU and win the engineering contest, you get a trip to Washington.

B19. If $x \ge 8$, then $x^2 \ge 64$ and if $x \le -8$, then $x^2 \ge 64$.

B21. If your earned income was more than $3700 you must file a return, and, if your unearned income was more than $1,300 you must file a return.

B23. If $x \in A\text{-}B$, then $x \in A$, and if $x \in A\text{-}B$, then $x \notin B$.

B25. [7 columns] (This is a very important theorem.)

A	B	C	$B \Rightarrow C$	$A \Rightarrow (B \Rightarrow C)$	$A \wedge B$	$(A \wedge B) \Rightarrow C$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Columns 5 and 7 prove it.

B27-29. [Use truth tables, or one wide truth table.]

H	C	$H \Rightarrow C$	not$(H \Rightarrow C)$	not C	$H \wedge$(not C)	$H \wedge (H \Rightarrow C)$	$H \wedge (H \Rightarrow C) \Rightarrow C$
T	T	T	F	F	F	T	T
T	F	F	T	T	T	F	T
F	T	T	F	F	F	F	T
F	F	T	F	T	F	F	T

B27. Requires 6 columns. Columns 4 and 6 prove it.

[Use a table]
B29.xx Requires 5 columns.
B31. Requires 6 columns.

B33. Answered on p. 155.
and - intersection, or - union, not - complement, if..., then ... - subset, iff - =
B35. See Figure 3.1.8. The region of H being inside the region of C is not the same as the region for C being inside the region for H.
B37. a) See Figure 3.1.2 and note that the set corresponding to A and B is a subset of the set corresponding to A. b) $S \cap T \subset S$.

B39. See Figure 3.1.8. b) The region for "(not H) or C" is everything. c) T3.1.12.
B41. Yes.
B43. Use a truth table with 9 columns.

^^^^^^^^
C1 & C2. It is not possible to assert that H causes or proves C just because "$H \Rightarrow C$" is true. For example, "1+1 = 2," is true and so is the Fundamental Theorem of Calculus, but "1 + 1 = 2" hardly causes or proves the Fundamental Theorem of Calculus.
When the components have variables (the subject of Chapter 4) then the word *causes* may be appropriate. For example, "For all x, $x > 5 \Rightarrow x^2 > 25$" is true. But "$x^2 > 25$" is not true by itself. It is true *when* $x > 5$, so the hypothesis does "cause" the conclusion to be true. However, it will still not *prove* it true. The conclusion is an open sentence and not always true, so it cannot be proven. We can prove "$H \Rightarrow C$" but not "C".
C3. Forms are abstract representations of statements in which the connectives are exhibited and the simple component sentences are represented by letters. On the other hand, statements have meaning as well as form. Forms are abstracted from examples of statements.
C5. a) "$A \Rightarrow B$" is LE to "not(A and (not B))."
b) "A or B" is LE to "not[(not A) and (not B)]."
c) "A iff B" is LE to "not(A and not B) and not(B and not A)."

C7. In mathematics, a definition fixes the meaning of a term and is, therefore, automatically true. However, after it is fixed it can be incorrectly reproduced. The statement which is false is so because it is not the (original, agreed-upon) definition.

C9. "If $x \in$ S and S $\cap$ T = S, then $x \in$ T."

Section 3.3. Logical Equivalences with a Negation

A1.

B	not B	$B \wedge$(not B)
T	F	F
F	T	F

A3. [9 columns]

A	B	C	$A \wedge B$	$A \wedge B \Rightarrow C$	not C	$A \wedge$(not C)	not B	$A \wedge$(not C)$\Rightarrow$(not B)
T	T	T	T	T	F	F	F	T
T	T	F	T	F	T	T	F	F
T	F	T	F	T	F	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	F	T	F	F	F	T
F	T	F	F	T	T	F	F	T
F	F	T	F	T	F	F	T	T
F	F	F	F	T	T	F	T	T

Columns 5 and 9 prove it.

A5. [8 rows and 8 columns]

A7. [8 rows and 8 columns]

A9. a) c $\leq$ 9 and c $\geq$ 1, which can also be written 1 $\leq$ c $\leq$ 9.
b) $x \leq$ -5 or $x \geq$ 8.

A11. a) $x \leq$ 25 and $x \geq$ 5, which can also be written 5 $\leq x \leq$ 25.
b) $x \leq$ 8 or $x \geq$ 32.

A13. True. Let C be a contradiction. $H \Rightarrow C$ is T iff H is false, by the definition of " $\Rightarrow$."

A15. In the "not both" version one or the other could be 0, but not in the "both not" version.
[e.g. a = 1 and b = 0 satisfies "not both 0" but does not satisfy "both not 0."]

A17. He is not dead and not in jail.

A19. She is not tall or not smart.

A21. If it is small and I cannot lift it, it is not light.
[or] If it is light and I cannot lift it, it is not small.

A23. When I go on vacation and do not go to the ocean, I go to the mountains.
[or] When I go on vacation and do not go the mountains, I go to the ocean.

A25. a) "The TV show is a sitcom" is A, "The TV show is in prime time" is B, and "The TV show makes lots of money" is C.
b) A and $B \Rightarrow C$. A and (not C) $\Rightarrow$ (not B).
c) B and (not C) $\Rightarrow$ (not A).

A27. Yes. $A \Rightarrow B$ and C. ($A \Rightarrow B$) and ($A \Rightarrow C$).

A29. a) sentence b) set c) function d) number

A31. a) set b) number c) sentence d) set

A33. $\cap$ connects sets, not sentences

A35. *and* connects sentences, not sets
A37. *and* connects sentences, not sets
A39. OK
A41. *or* connects sentences, not sets
A43. $\cap$ connects sets, not sentences
A45. *or* connects sentences, not sets, and $(8,\infty)$ is a set.
A47. *not* applies to sentences, not expressions
A49. OK
A51. *or* connects sentences, not sets
A53. awkward, but OK
A55. Would be OK if "S" were a sentence, but it is usually a set.
A56. OK
A57. numbers and sentences cannot equal sets
A59. members are written in lower case letters
A61. OK
A63. OK
A65. OK
A67. $\Rightarrow$ connects sentences, and "x" is an expression.
A69. "A" should be lower case
A71. should be lower case letters
A73. $\Rightarrow$ connects sentences, and "$S \cap T$" is an expression.
A75. OK
A77. or
A79. a) neither, it is just true b) neither, it is just true c) tautology d) neither
A81. A is "$x \in$ S," B is "$x \in$ T."

^^^^^^^^

B1. The negation of "$H \Rightarrow C$" is "H and not C." The original says, if H is true, then so is C.
The negation says that is false, so H must be true but C not.
The original is false iff H is true and C is not.
The original says nothing about C if H is false, so the original cannot be false if H is false.
B3. You must check E (the letter is a vowel)
and 7 (the number is not even, the hypothesis of the contrapositive)
B5. You must check K (the letter is a consonant)
and 5 (because the number is not even, the hypothesis of the contrapositive)

B7. If $xy > 25$ and $|x| \le 5$, then $|y| > 5$.
B9. Pro basketball players who are not tall are quick.
B11. If $0 < x$ and $x^2 \ge z^2$, then $x \ge z$. [or] If $x < z$ and $x^2 \ge z^2$, then $x \le 0$.
B13. If $c < 0$ and $ca \le cb$, then $a \ge b$.
B15. Fords that are not great trucks do not have four-wheel drive.
B17. a) If $x + y > 2$ and $x \le 1$, then $y > 1$.
b) If $x \le 1$ and $y \le 1$, then $x+y \le 2$,
B19. a and c. b, d and e.

B21. The book is not rare and the book is old.
The book is not both rare and old.
The second, The book is not (rare and old).
B23. $A \Rightarrow (B \text{ or } C)$ is LE to $A \Rightarrow (\text{not } B \Rightarrow C)$ [by "or"]
is LE to A and not $B \Rightarrow C$ [by a Hypothesis in the Conclusion]
B25. Use a truth table, or better, substitute "$H \Rightarrow C$" for "A" and simplify with 3.3.2.
B27. "not(B or C) $\Rightarrow$ not(not A)" is LE to "(not B) and (not C) $\Rightarrow A$" by DeMorgan's law and double negation.

B29. It is not a contradiction. If "A is false, then both "$A \Rightarrow B$" and "$A \Rightarrow$ (not B)" are true.

B31. True. [Use a truth table or, better, note the Theorem on Cases.]
B33. a) Could sketch something like Figure 3.1.3 and label the outer region empty. ($\varnothing$)
b) (not B) $\Rightarrow C$ (since the outer region is empty). Theorem 3.3.5.
B35. Sketch 5 figures. A or B (Fig. 3.1.3), not(A or B), not A, not B, (not A) and (not B).

B37. (B or C) iff (not B) $\Rightarrow C$ [is a tautology]
B39. not complement, and $\cap$, or $\cup$, if...then $\Rightarrow$, iff $=$
B41. $R \subset (S \cap T)$ iff $R \subset S$ and $R \subset T$.

^^^^^^^^
C1. If "not A" implies a contradiction, it must be false, in which case its negation is true. So A must be true.
C3. D and E are true and false together, that is, for the same combinations of truth values of their simple components. Therefore, any more complex statement built up from D with connectives arranged in any particular order would have the same columns as the a more complex statement built up from E with connectives arranged in the same particular order.

Section 3.4. Tautologies and Proofs

A1. A proof of the contrapositive proves the statement because the contrapositive is logically equivalent to the statement and has the same truth value as the statement. Therefore, a proof that one is true proves the other is true.
A3. T A5. T
A7. A and B A9. $B \Rightarrow$ (not A)
A11. $A \wedge B \wedge$(not C) A13. It is LE to "$A \Rightarrow B$ and $A \Rightarrow C$."
A15. $A \Rightarrow B$ iff C A17. $H \Rightarrow \mathrm{D} \Rightarrow \mathrm{E} \Rightarrow \mathrm{F}$;
A19. [Your choices] a) $3x > 15$, $x > 5$. b) $x^2 + 3 > 0$.
A21-24. [Use truth tables. xx]
A25. not(not B) iff B.
A26-28. Connect the LE statements by iff.
A29. If $x \in \mathrm{S}$, then $x \in \mathrm{T}$. A31. $x \in \mathrm{S}$ iff $x \in \mathrm{T}$.

^^^^^^^^
B1. Only d. The contrapositive)
B3. Yes. B5. No. B7. No.
B9. True. $A \wedge B \Rightarrow A \Rightarrow C$, so $A \wedge B \Rightarrow C$.
B11. True. [The contrapositive.]
B13. True. $A \Rightarrow (A$ or $B) \Rightarrow C$. Thus $A \Rightarrow C$.
B15. a) nothing b) John is quick. c) Nothing very firm, but we can deduce:
If he is not quick, he is not a pro basketball player, which is logically equivalent to: He is quick or not a pro basketball player. (by 3.1.12) [or] if he is a pro basketball player, he is quick (by contrapositive).
d) nothing firm. e) John is not a pro basketball player. f) nothing
B17. a) nothing very firm, but we can deduce: If it has four-wheel drive, it is a great truck.
[or] Either it does not have four-wheel drive or it is great
[or] If it is not a great truck, it does not have four-wheel drive.
b) The truck does not have four-wheel drive. c) nothing
d) nothing e) The truck is either not a Ford or not four-wheel drive.

B19. a) nothing very firm, but we can deduce: If it is in prime time, then it makes lots of money [or] If it does not make lots of money, then it is not in prime time [or] Either it makes lots of money or is not in prime time.
b) nothing c) It is not a sitcom in prime time. d) It is not a sitcom.

B21. a) $y < 4$. b) nothing c) nothing d) $x \le 3$.

B23. a) nothing b) $x \le 5$ c) nothing d) $y > 9$

B25. [(not B)$\wedge$(not C) $\Rightarrow$ (not A)]. Solution by DeMorgan's:
[not(B or C) $\Rightarrow$ (not A)] iff [(not B)$\wedge$(not C) $\Rightarrow$ (not A)]

B27. a) not[(A or B) $\Rightarrow$ C] iff not[($A \Rightarrow C$)$\wedge$($B \Rightarrow C$)]
iff [not($A \Rightarrow C$)] or [not($B \Rightarrow C$)] iff [$A\wedge$(not C)] or [$B\wedge$(not C]
b) Yes. $A\wedge$(not C) would do.

B29. $A\wedge B\wedge C \Rightarrow$ D is LE to $A\wedge B\wedge$(not D) $\Rightarrow$ (not C)

B31. a) $H \Rightarrow C$ b) (not C) $\Rightarrow H$ c) (not C) $\Rightarrow$ (not H)
d) (not H) $\Rightarrow$ (not C) e) (not C) or H f) (not H) or C
a, c, and f are LE. Also, d and e

B33. a) $H \Rightarrow C$ b) $C \Rightarrow H$ c) (not H) $\Rightarrow$ (not C)
d) (not C) $\Rightarrow$ (not H) e) (not H) or C f) (not C) or H
a, d, and e are LE. Also, b, c and f are LE.

B35. 1 and 7 follow.

B37. 1, 2, 6, and 7 follow.

B39. 4 and 8 follow.

B41. "$A \Rightarrow B \Rightarrow C$" is an *abbreviation* for "$A \Rightarrow B$ and $B \Rightarrow C$."
The other two are not LE to it, or to each other.

B43. a) If H, then (if $x \in$ S, then $|x| \le 120$). b) $H \Rightarrow (A \Rightarrow B)$
c) Theorem on a Hypothesis in the Conclusion.
d) A, that is, "$x \in$ S."
e) B, that is, "$|x| \le 120$."
f) H, or prior results

B45. $S \subset S \cup T$. B47. $(S^{c})^{c} = S$. B49. $S \subset T$ iff $T^{c} \subset S^{c}$.

B51. $R \subset (S \cap T)$ iff $R \subset S$ and $R \subset T$. B53. $R = S$ and $S = T \Rightarrow R = T$.

B55. $R \subset (S \cup T)$ iff $(R \cap S^{c}) \subset T$.

B57. For an assignment of truth values to the simple components, the truth values of A and B are equal by the first hypothesis, and the truth values of B and C are equal, by the second. By the transitivity of equality, which is axiomatic, the truth values of A and C are equal. Thus A and C are logically equivalent. ■
Abbreviated proof: A has the same truth value as B, by the first hypothesis, and B has the same as C, by the second. Therefore A has the same truth value as C. ■

^^^^^^^^

C1. a) No. For example, 1+1 = 2 does not imply S is a subset of S union T.
b) Yes. "Imply" is used to abbreviate "if..., then..." and has no causal implication in truth tables (although it will when hypothesis and conclusion contain the same variables, as in Chapter 4).

C3. $A \Rightarrow B\wedge C$ (hypothesis)
$\Rightarrow B$ (Tautology 3.4.3)
Therefore $A \Rightarrow B$ (transitivity of " $\Rightarrow$ ")

C5. A and $B \Rightarrow C$. Proved in the same form.

C7. $A \Rightarrow (B \Rightarrow C)$ [or] (A and B) $\Rightarrow C$. Proved as A and not $C \Rightarrow$ not B.

C9. $A \Rightarrow B$ or C. Proved as not(B) and $A \Rightarrow C$, by "or" in the conclusion.

C11. A and $B \Rightarrow C$ [or $A \Rightarrow (B \Rightarrow C)$] Proof: B and $A \Rightarrow C$.

Chapter 4. Sentences, Variables, and Connectives

Section 4.1. Sentences with One Variable

A1. Open sentences, generalizations, and existence statements.
A3. The true generalizations are: b, c (they use placeholders) [*f* is free, it is being defined]
A5. The true generalizations are: a, c (they use placeholders)
A7. The true generalizations are: b (it uses a placeholder)
A9. The true generalizations are: b (it uses a placeholder)
A11. The true generalizations are: a, c (they use placeholders)

A13. Those fitting the convention are: b,c
A15. Those fitting the convention are: a,c
A17. b, d

^^^^^^^^

B1. A variable which is quantified (by "for all" or "there exists") is a placeholder (bound variable). A variable which is not quantified is free (and the sentence is an "open" sentence). The truth of an open sentence with a free variable may depend upon which element of the universal set the variable represents. However, a sentence with a placeholder is simply true or false (not depending upon which element of the universal set the variable represents). Also, the truth of a sentence with a placeholder does not depend upon which letter represents that variable (but certain conventional usages should be respected). Free variables refer to particular elements in the universal set, such as numbers. Placeholders are used to discuss higher-level concepts such as operations and order.

B3. An equation is an open sentence -- the truth may depend upon the value of the variable. An identity is a special type of equation, one which is true for all values of the variable. If an equation is true "for all x," it is an identity (and interpreted as a generalization). Usually, equations are about numbers and identities are about operations and order.

B5. a) No. b) Yes.

B7. a) S b) S, T c) a, b, c

B9. a) generalization b) open c) existence statement
d) open e) generalization
Generalizations and existence statements use placeholders. Open sentences use free variables.

B11. f is defined by this, for all z. (placeholder)
f, previously defined, determined to be equivalent to this. (placeholder)
f, previously defined, set equal to this (thinking to solve for z). (free)

B13. a) As an equation (open sentence), true for some z and false for others.
b) It's not an identity.
c) This is an identity, and is true for all z. It is about multiplication and subtraction.

B15. y [If it were in "sentence-form" with two equivalent sentences, all the variables would be placeholders.]

B17. a) no. b) connectives and order. c) placeholder

^^^^^^^^

**** C1-21 have **many possible answers**, of varying degrees of complexity. These are only examples of possible answers, and many would likely not be selected by students on their own.

C1. $x - 6 < x$.
C3. $S \cap T \subset S$.
C5. $(x + 1)^2 = x^2 + 2x + 1$.
C7. $S \subset S$.
C9. $2x < z$ iff $x < z/2$
C11. If $c \neq 0$, $cx = d$ iff $x = d/c$

C13. $10x = b$ iff $x = b/10$. C15. $\{2\} \subset \{2,3,4\}$ C17. $a = b$ iff $b = a$ ("=" is symmetric).

C19. "$\Rightarrow$" is transitive.

C21. (A or B) $\Rightarrow C$ implies $A \Rightarrow C$.

C23. a) It could be quantified by "For all x, c, and d" or "For all c and d."
b) The latter would be a statement asserting the equivalence of two open sentences in x, the former would be a statement.
c) The two versions have the same meaning. Nevertheless, the former (with x) helps concentrate attention on the equation-solving use of the theorem with x on the left.

Section 4.2. Generalizations and Variables

A1. a) TG b) TG c) O

A3. a) O b) TG c) O

A5. a) P b) P c) F

A7. a) F b) P c) F

A9. a) b b) b, c A11. a) x, c b) a, b, c

A13. a) F (Consider negative z) b) F (Consider false B)
c) T d) T

A15. a) $d > 0$ and $b > c \Rightarrow bd > cd$. b) Solve $2x = 5\text{-}3x$.
c) $3c = 4c^2 - 90$.
d) If A is false, then $A \Rightarrow B$. [Alternative: not $A \Rightarrow (A \Rightarrow B)$]
e) $A \wedge (A \Rightarrow B) \Rightarrow B$. alternative: "$A \wedge (A \Rightarrow B) \Rightarrow B$" is a tautology.

A17. a) $S \subset S \cup T$. b) $S \subset S$. c) $(x\text{-}1)^2 = x^2 - 2x + 1$.

A19. 1 A21. 2 A23. 2 A25. 3

^^^^^^^^

B1. [Answered in Section 4.1.] Otherwise generalizations could be made false when the hypotheses didn't apply!

B3. See p. 26, bold. Identities are about operations and order. They give alternative ways to evaluate a sequence of operations. Other equations are about numbers (particular numerical values).

B5. a) numbers b) operations on equations and inequalities

B7. a) set b) set theory concepts c) sets

B9. a) number b) function

B11. r is an independent variable, A a dependent variable

B13. x is an independent variable, y a dependent variable

B15. x is an unknown, c is a parameter.

B17. This defines c in terms of the previously-given a.

B19. a and b are placeholders.

B21. x is an independent variable, y a dependent variable, and c is a parameter

B23. x is quantified, c is a parameter, and this defines f.

B25. This defines T in terms of the previously-known S.

B27. x is an independent variable, y a dependent variable, and k a parameter.

B29. x is an independent variable, y a dependent variable, and r is a parameter
B31. x is an independent variable, y a dependent variable. m and b are parameters.
B33. If $x > 0$ and $a < b$, then $ax < bx$.

B35. Add in "and b $\neq$ 0." to the hypothesis. That is "If bc = bd and b $\neq$ 0, then c = d."
B37. For example, $y = mx + b$ uses m and b as parameters.
B39. a) x is the independent variable, y the dependent variable in the definition of a functional relationship. m and b are parameters.
b) Traditional usage.
c) No. It is a definition of the relationship between y and x.
B41. a) The variables are a, b, c, and x. The first three may also be regarded as parameters (constants). b) Usually "For all a, b, and c," but "For all a, b, c, and x" would be true too, but less appropriate. c) a, b, and c.
B43. a) $\{(x, x^2)\}$ [or] $\{(x, x^2) \mid x \text{ is real}\}$
b) $\{(x, y) \mid y = x^2\}$
**** B44-64. [There are many possible answers to each problem.]

B45. $10(x + 2) = 10x + 20$	B47. $S \subset S \cup T$.	B49. $(x + 50)/10 = x/10 + 5$.
B51. $(S^c)^c = S$.	B53. $x - a = b$ iff $x = b + a$.	B55. $x + 5 > x$.
B57. $\lvert x\rvert > c$ iff $x < -c$ or $x > c$.	B59. $a < b$ and $b < c \Rightarrow a < c$.	B61. *iff* is transitive.

B63. not(not B) is LE B.

^^^^^^^^
C1. The student appears to be erroneously concentrating on the conclusion, when he should be evaluating the whole statement. You should point this out to him. You might say that the statement <u>is</u> always true, since any statement is regarded as true whenever the hypothesis is false. Thus exceptions can never be found when the hypothesis is false.
Alternative: A conditional statement is true if the hypothesis is false (by definition of "if..., then..."). Therefore it <u>is</u> true when $b = 0$.
C3. a) A point (ordered pair) is marked on the graph of an open sentence with two variables when the sentence is true and not marked if it is false.
b) The graph of f is the graph of the open sentence in two variables "$y = f(x)$."
[or] The set of (x, y) such that a sentence in those two variables is true is a graph.
The set of $(x, f(x))$ for all x is a graph. It is the same graph as the set of ordered pairs that make $y = f(x)$ true.
C5. If we consider the set of all ordered pairs, $\{(x, y)\}$, we can solve an equation in two variables (such as "y = 2x") for the pairs that make it true. So, the single variable, the pair (x, y) serves as a "free" variable in "$y = 2x$." Now, in this higher-level space, we can think of functions as objects. Then, there is a solution (for y) for each x. So we can say, "For all x, $y = 2x$."

Section 4.3. Existence Statements and Negation

A1.	a) An existence statement.		b) A generalization.	
A3.	a) 1 and 4	b) 3 and 4	c) 3 and 4	
	[or] 1) a	2) none	3) b, c	4) a, b, c

Note to problems A5 - A12: Subscripts may be included or omitted. Also, as always, quantified variables may be represented by other letters.
A5. There exists x such that $g(x) > 12$.
A7. There exists $x \in S$ such that $x > 23$.
A9. There exists $x \in S$ such that $x < 45$.
A11. There exists x such that $\lvert x\rvert > 7$ and $x \leq 7$.

A13. Let x = -1.
A15. x = -10
A17. x = -1
A19. x = -20
A21. a = 0 and b = -2.
A23. Let x = 1. Then $x^2 - 2x + 1 = 0$, not > 0.
A25. Let $S = \{0, 1\}$ and $T = \{2\}$.

A27. a) "Upper bound" is a noun. b) "Bounded above" is an adjective.
A29. "Not all batteries are alike."
A31. a) T b) F c) F d) T
A33. False. Let $S = (1, 3)$ and $T = (2, 5)$. [or] S = {1,2} and T = {1,3}.
A35. There exist x and y such that $x^2 > y^2$ and $x \leq y$.

^^^^^^^^
B1. No. It could be that some (but not all) are and some (but not all) are not.
B3. a) There exist a and b such that $ab > 0$ and not($a > 0$ and $b > 0$).
[or] There exist a and b such that $ab > 0$ and $a \leq 0$ or $b \leq 0$.
b) Let $a = b = -1$. Then $ab = 1 > 0$.
B5. a) There exists an element of S with value greater than 565.
[There exists $x \in S$ such that $|x| > 565$.]
b) Let x = 800. $\sqrt{800} < 30$ but 800 > 565.

B7. F. $5.5 \in$ the set.
B9. F. x = -5 and y = 0.
B11. F. x = 0 and b = -5.
B13. F. b = -5 and c = 0.
B15. There exists a rational number which is not an integer.
B17. a) There exist x and z such that $x < z$ and $f(x) \geq f(z)$. b) Let x = -1 and z = 0.

B19. a) There exists x such that $x \in T^c$ and $|x| \geq 20$. b) $(-\infty,-20]\cup[20,\infty)$

B21. a) "For all x, $x^2 + 1 > 0$." b) "There exists x such that $f(x) = 0$."
B23. False. Let a = 5, b = 6, c = 0, d = 2. Then $a\text{-}c \geq b\text{-}d$.

B25. "There exist x and z such that $x < z$ and $f(x) > f(z)$."
Alternative: "There exist $x < z$ such that $f(x) > f(z)$."
B27. The former is always true and can be regarded as an abbreviation omitting "For all S and T." The latter is not always true and describes S and T with an open sentence.

B29. a) F b) T B31. a) F b) T B33. a) yes b) no
B35. a) For all x there exists y such that $x < y$. b) Yes, the negation is true.
B37. a) There exists $x > 0$ such that for all $y > 0$, $y \geq x$. b) no. The original is true.

B39. No piles have more than one ball [or] All piles have at most one ball.
B41. All horizontal lines intersect the graph at most once.
B43. All vertical lines intersect the graph at most once.
B45. Y B47. Y B49 Y B51. N
B53. Y B55. N B57. N B59. Y

B61. It depends upon where the "For all x" is. It could be on the whole, in which case it is true. It could be on each component, in which case it might not be true.

B63. All zaps are not wrets. B65. All zaps are wrets.
B67. No zaps are wrets, All zaps are not wrets. B69. nothing B71. nothing

^^^^ Note: “Some a’s are b’s” implies “some b’s are a’s”, but that will not be mentioned here.
B73. Some zaps are not wrets. [or] Some wrets are not zaps.
B75. Nothing. B77. Some ubes are not wrets.
B79. Nothing. B81. Nothing.
B83. Some wrets are not zaps.

^^^^^^^^
C1. n is divisible by 3 iff there exists j such that $n = 3j$.
C3. y is inversely related to x iff there exists $k \neq 0$ such that $y = k/x$.
C5. There exists x such that H(x) and [(x $\in$ S and x $\notin$ T) or (x $\in$ T and x $\notin$ S)].
C7. There exists x such that $H(x)$ and $x \in S$, and $x \notin T$ and $x \notin R$.
C9. b is a lower bound of S iff, if $x \in S$, then $b \leq x$.
C11. not(A and B) is LE to (not A) or (not B).
"All" is like "and" but for possibly more than 2.
"There exists" is like "or" but for possibly more than 2.
C13. There exists M such that for all N there exists n
such that $n > N$ and $a_n \leq M$.
C15. Let $f(x) = 2x$ C17. Let $f(x) = 2^x$. C19. Let $f(x) = x^3$.

Section 4.4. Ways to State Generalizations

A1. a) Let x be negative, say, $x = -1$. b) Let x be positive, say, $x = 1$.
c) Every number (and $f(x)$ represents a number) is either positive or non-positive.
[The point is, "or" can be misused in combination with "for all"]
A3. If a number ends in 0, then it is divisible by 10.
A5. For all sets S, $\varnothing \subset$ S.
A7. “If p is prime and p divides ab and p does not divide a, then p divides b.”

^^^^^^^^
B1. Many generalizations can be expressed using "if..., then..." and omitting the implicit "For all ...". Readers are supposed to know that conditional sentences are usually interpreted as generalizations. Some generalizations can be expressed using plurals. The use of the plural is the indication it is a generalization.
B3. Let $f(x) = 1$ if $x \geq 0$, and 0 if $x < 0$. Let $g(x) = 0$ if $x \geq 0$ and 1 if $x < 0$.
Then $f(x)g(x) = 0$ for all x.
B5. a) "If two integers are consecutive, then their sum is odd."
b) "i + (i+1) is odd." or "If i is an integer, then i + (i+1) is odd."
or "If i and j are consecutive integers, then i + j is odd."
B7. a) "If two numbers are positive, then their sum is positive."
b) "If a > 0 and b > 0, then a + b > 0."
c) "If $a + b \leq 0$, then $a \leq 0$ or $b \leq 0$."
B9. An existence statement. "There exist two integers ending in 5 such that their sum is 100."
[By the way, it is true: 45+55 = 100.]
B11. A generalization. "If a > 0 and b > 0, then ab > 0."
B13. [For all integers i] If i is divisible by 6, then i is even.
B15. [This is false as a generalization, so it must be an existence statement] "There exists $x > 0$ such that $x^2 < x$.

B17. Math classes are fun.
B19. It could mean *all*, as in $S \subset T$. Or it could mean *some*, as in $S \cap T \neq \varnothing$.
B21. a) If a rectangle has perpendicular diagonals, then it is a square.
b) If a figure is not a square, then it is not a rectangle with perpendicular diagonals.
[or] If a rectangle is not a square, its diagonals are not perpendicular.]
B23. A generalization. $S \cap T = \{x | x \in S \text{ and } x \in T\}$.
B25. If $a+b$ is not positive, then a and b are not both positive.
If $a+b$ is not positive, then $a \leq 0$ or $b \leq 0$.
B27. If ab is not positive, then a is not positive or b is not positive.
B29. If $b < 0$ and $c > 0$, then $bc < 0$.
B31. False. 36 is divisible by 18 and 12, but not by (12)18.
B33. Let $a = 1$, $b = 0$ and $c = 2$. Then it says $x^2 + 2 = 0$, which is never true for real numbers.
B35. a) If S is a closed set, then S^c is an open set. b) If S^c is not open, then S is not closed.

^^^^^^^^

C1. The symmetric difference of S and T is $(S \cap T^c) \cup (T \cap S^c)$. [or $(S \cup T) \cap (S \cap T)^c$]

Section 4.5. Reading Theorems and Definitions

A1. The sentence being defined, its definition, and the "definition in sentence-form," which is a compound sentence.
A3. a) $1 + 2 + 3 + \ldots + n$ b) $n(n + 1)/2$.
A5. 200(201)/2 = 20100
A7. 1000(1001)/2 = 500,500
A9. $x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-5)}}{2(1)} = \dfrac{-3 \pm \sqrt{29}}{2} = 1.19 \text{ or } -4.19$
A11. $x = \dfrac{-1 \pm \sqrt{1^2 - 4(-1)(4)}}{2(-1)} = \dfrac{-1 \pm \sqrt{17}}{-2} = -1.56 \text{ or } 2.56$
A13. a) $x \in S^c$ b) No, it is open. c) The definition of it is open
The definition in sentence-from is true (by definition).
A15. a) b is a bound of S. b) noun
A17. S is a subset of T iff if $x \in S$, then $x \in T$.
A19. c is an upper bound of T iff if $y \in T$, then $y \leq c$.
A21. a) true b) false
A23. If $x \in (1,3)$, then $x \leq 35$.
A25. If $x \in [-45,23)$, then $|x| \leq 50$.
A27. 3 is one such upper bound. Any b, $2 \leq b < 5$ would do.
A29. a) $[3, \infty)$ b) $[3, \infty)$ c) $[4, \infty)$ d) Yes, they are *in* the set of upper bounds.
A31. 10 in an upper bound of (-20, 9) but not a bound of it.
A33. a) no b) yes c) no d) yes e) yes f) no g) no
A35. If $x < z$, then $x^3 < z^3$
A37. $[3(x+h)+5 - (3x+5]/h = 3h/h = 3$ if $h \neq 0$.
A39. $[-5 \pm \sqrt{(25 - 4(1)(-2))}]/2$
A41. $[-7 \pm \sqrt{(49 - 4(-2)(12))}]/(-4)$

A43. $|y| < b$ iff $-b < y < b$.
A45. T2.1.12: R is a subset of P iff if $y \in$ R then y $\in$ P.
["S is a subset of T iff if y $\in$ S, then y $\in$ T." Note: Only the "x" can be changed in a definition of "S $\subset$ T."]

A47. If $x < z$, then $\sqrt{x} < \sqrt{z}$ (if x and z are in the domain, that is, $x \geq 0$).
A49. $|x| \geq 5$ or $x^2 < 25$.
A51. a) no b) connectives. [the relationship of *not* and $\Rightarrow$ (in a certain order).]
c) placeholder
A53. (not C) $\Rightarrow$ not(A and B);
A55. (not A) $\Rightarrow$ not(not B) which is LE to (not A) $\Rightarrow$ B.
A57. $A \wedge$not$(B \wedge$C)
A59. $A \wedge$not(B iff C).
A61. [several answers are possible] ($B \Rightarrow$ C) and ($B \Rightarrow A$)
A63. ($B \Rightarrow$ D) and (C $\Rightarrow$ D)
A65. not C and not B
A67. ($H \Rightarrow B$) and ($B \Rightarrow H$)
A69. $H \wedge$(not C)
A71. C and not $A \Rightarrow B$
A73. (C $\Rightarrow A) \wedge$(C $\Rightarrow B$)

^^^^^^^^

B1. A mathematical definition in sentence-form defines an open sentence containing the term with another open sentence. The two sentences are equivalent and connected by "iff".
B3. Math defines sentences containing terms, not just the terms, so that logic can apply. Also, terms require a context and a sentence provides the context.

B5. Because math defines sentences containing terms, not just the terms. Terms require a context, which is given by including them in a sentence.

B7. b, d, e, f.
B9. a) bounded above, bounded b) bounded above (not bounded) c) neither
B11. a) Yes. b) No.
B13. $x = [-2b \pm \sqrt{(4b^2-4c)}]/2$
B15. $x = [-a \pm \sqrt{(a^2-4cb}]/2b$
B17. $x = [-b \pm \sqrt{(b^2 + 4ac)}]/2a$
B18. $y = [-3x \pm \sqrt{((3x)^2 - 4(1)(x^2 - 14))}]/2$
B19. $y = [8x \pm \sqrt{((-8x)^2 - 4(1)(x^2 + x - 20))}]/2$

B21. a) $j(j+1)/2$. b) $(k+1)(k+2)/2$.

B23. Larger arguments of f always yield larger images.
[Maybe the variable approach isn't easier here, but it can be more precise. For example, it could distinguish $\geq$ from $>$, if we wanted to say "at least as large as."]
B25. n is even implies there exists k such that $n = 2k$. So $n^2 = (2k)^2 = 2(2k^2)$ is even, by the definition of even.
B27. r is a rational number iff there exist (integers) i and j with j $\neq$ 0 such that r = i/j.
B29. $4.4 \in (4.2, 4.5) \subset [4.2, 4.5]$.
B31. False. If S is bounded, then S^c is not. Proof: There exists b such that $x \in S$ satisfies $|x| \leq b$, by hyp. So if $|x| \geq b$, then x is not in S. Therefore, for any d, there exists $x \in S^c$ such that $|x| > d$, so S^c is not bounded.
B33. [There are 3 from intersection, and two from =]
1) If $x \in S$ and $x \in T$, then $x \in S \cap T$.
2) If $x \in S \cap T$, then $x \in S$. 3) If $x \in S \cap T$, then $x \in T$.
4) If $x \in R$, then $x \in S \cap T$. 5) If $x \in S \cap T$, then $x \in R$.
B35. A set is determined by its members, not how many times they are listed. That's just the definition of set equality. If you want to list things more than once, fine, but that is a different concept with a different name.

B37. if. only if.

B39. a) f is decreasing iff $x < z \Rightarrow f(x) \geq f(z)$. b) False. x^2 is neither.

B41. Is f given and fixed, or not? Apparently it is given. There exists an x such that H holds but $f(x) \leq 0$."
If this were about functions in general, denoted by f, the statement would say that all f that satisfy H have some property. In that case, the negation would be "There exists an f such that H holds but there exists x such that $f(x) \leq 0$."

B43. $5^* = 2$. $8^* = 4$. Thus the problem is to solve $n^* = 2(4) = 8$. The solution is $n = 16$ or $n = 17$.

B45. $n^* = (3^*)(6^*) = 9(3) = 27$, so $n = 9$ or $n = 54$.

B47. $(-4)' = 12$. $2' = 4$. So the equation is $n' = 48$. $n = 24$ or -16.

B49. $f(5) = 15$. $f(14) = 7$. $f(n) = 15(7) = 105$. $n = 35$ or 210.

B51. For all $c > 0$ and all b, if, for all x, f is defined by $f(x) = c\mathrm{x} + b$, then f is increasing.

B53. $B \wedge H \Rightarrow \mathrm{C}$ a Hypothesis in the Conclusion

B55. $B \wedge \mathrm{C} \wedge \mathrm{D} \Rightarrow \mathrm{E}$ "or" in the conclusion and double negation.

B57. a) H and not C $\Rightarrow$ not A.
b) not C $\Rightarrow$ not(H and A). [Could be followed with DeMorgan's law]

B59. $[A \Rightarrow (B \wedge \mathrm{C})]$ and $(A \Rightarrow \mathrm{D})$ $[A \Rightarrow B]$ and $[A \Rightarrow (\mathrm{C} \wedge \mathrm{D})]$

B61. $-3\pi \sin(\pi x)$ B63. $14 \cos(2x)$

B65. $(3/4)\cos(x/4)$ B67. $15x^{4}$

B69. $(2.3)12x^{1.3} = 27.6x^{1.3}$

B71. $x \in S\text{-}T$ iff $x \in S$ and $x \notin T$.

B73. b is a lower bound of S iff, $x \in S \Rightarrow b \leq x$.

B75. a) {1,2} b) {6,7}

B77. 120 B79. $x = 20$.

B81. -2

B83. $8a - 2bc$

B85. $15 - 2x = 3$. $x = 6$

B87. $x^2 - 16 = 9$. $x = \pm 5$

B89. a) (6, 8, 0) b) $(d, a, 0)$ c) $(c, 0, 0)$

B91. $50^2 = 2500$

B93. Let $k = 2n - 1$. $n = (k + 1)/2$. Sum is $((k + 1)/2)^2$.

^^^^^^^^

C1. [Any new symbol will do. The conventional symbol is Δ]
$x \in \mathrm{SdT}$ iff $x \in \mathrm{S} \cup \mathrm{T}$ and $x \notin \mathrm{S} \cap \mathrm{T}$.
Alternative: $x \in \mathrm{SdT}$ iff ($x \in \mathrm{S}$ and $x \notin \mathrm{T}$) or ($x \in \mathrm{T}$ and $x \notin \mathrm{S}$).
Alternative: $\mathrm{SdT} = (\mathrm{S} \cup \mathrm{T}) \cap (\mathrm{S} \cap \mathrm{T})^c$.

C3. f is a linear function iff there exists k such that $f(x) = kx$.
[Note: Vertical lines are not considered "linear" functions.]

C5. $[-2x \pm \sqrt{(4x^2 - 4x^4)}]/2x$ [if $x \neq 0$].

C7. Treat x as $(\sqrt{x})^2$. $\sqrt{x} = [-1 \pm \sqrt{(1+400)}]/4$. The negative is impossible as a square root, so $x = \{[-1+\sqrt{(1+400)}]/4\}^2 = 22.6$.

C9. a) 1000(1001)/2 - 99(100)/2.
b) $n(n+1)/2 - (k-1)k/2$ [I'd stop here.]
$= (n^2 + n - k^2 + k)/2 = (n^2 - k^2 + n + k)/2 = (n - k + 1)(n + k)/2$.

C11. a) $1000^2 = 1{,}000{,}000$.
b) $1 + 3 + 5 + ... + (2n - 1)$ is n^2, by inspection, which is not a proof.
It is all integers up to $2n - 1$ minus the even ones
$1 + 2 + 3 + 4 + ... + (2n - 1)$ - (a sum of n-1 even integers)
$= (2n - 1)(2n)/2 - (n - 1)(n)$ [from C10]
$= n(2n - 1 - (n - 1)) = n^2$. [This can be also proved by mathematical induction.]
c) $[(k+1)/2]^2$, because $k = 2n - 1 \Rightarrow n = (k+1)/2$.

C13. Trivial: Let $N = (0, 1)$. Then, if $x \in (0,1)$, $x \in (0, 1) \subset (0, 1)$.

C15. There is $k \neq 0$ such that $y = kx$. There is $c \neq 0$ such that $z = cy$. Therefore, $z = c(kx) = (ck)x$, where $ck \neq 0$.

C17. There exists M such that for all X there exists $x > X$ such that $f(x) \leq M$.

C19. a) $m = \sup S$ iff (1) $x \leq m$ for all $x \in S$ and (2) if $x \leq b$ for all $x \in S$, then $m \leq b$.
b) 3 c) 3 d) 4 e) -2

C21. 1*2 = 6. 2*2 = 4. 4*1 = 1. So the right side is 24.
x*24 = 24. So the solution is $x \geq 24$ (Only the first case yields solutions.)

C23. Let $S \in P(S)$. So, by hyp, $S \in P(T)$. So $S \subset T$ by the def of the power set of T.

Section 4.6. Different Appearance -- Same Meaning

A1. a) false b) true c) true

A3. a) true b) true c) false

A5. a) false b) false

A7. a) F b) T

A9. a) F (but close. T if $c \neq 0$.) b) T

A11. a) F b) F [$\sqrt{-7}$ does not exist]

A13. a) T b) T

A15. F. Let $S = \{1, 2\}$ and $T = \{2\}$.

A17. a) If $x \in$ T, then $|x| \leq 5$. b) If y $\in$ T, then $|y| \leq 5$.
c) If $|x| > 5$, then $x \notin$ T. (contrapositive of (a))

A19. $-2 < y+4 < 2$, using T1.5.5.

A21. If $x > 2$ and $f(x) > 0$, then $f(x)g(x) > 7$ [There are others.]

^^^^^^^^

B1. 1) "Definitions" (a definition in sentence-from asserts that two sentences are equivalent).
2) "Quantified variables" (placeholders may be replaced by other letters).
3) "Logical equivalences" (a sentence may be replaced by another logically equivalent to it).
4) "Theorems asserting equivalence" (a theorem may assert that two sentences are equivalent).
5) "English statements without variables" (the language of mathematics can be avoided).

B3. 1) If $x \in$ S, then $x \in$ T (definition)
2) If y $\in$ S, then y $\in$ T (letter-switching, quantified variables)
3) If $x \notin$ T, then $x \notin$ S (logical equivalence)
4) If $x \in T^c$, then $x \in S^c$ (theorem)
5) All the elements of S are in T (English)

B5. a) A or $B \Rightarrow$ C b) $(x > 5 \Rightarrow |x| > 5)$ and $(x < -5 \Rightarrow |x| > 5)$

B7. If $x \notin T$, then $x \notin S$.

B9. If $x > 9$, then $x \notin S$.

B11. There exists $x \in$ S such that $x \notin$ T.

B13. There exists $x \in$ S such that $x > 9$.

B15. a and c. b and d. e) b and d are true.

B17. c and f. b, d and e.
B19. All "no" except C8.
B21. converses: C2 & C8. C3 & C7

B23. contrapositives: theorem & C3
B25. b & d & e. a & c & f.
B27. There exists x such that ($x \in$ S and $x \notin$ T or $x \in$ T and $x \notin$ S).
Alternative: There exists x such that ($x \in$ S and $x \in$ T^c or $x \in$ T and $x \in$ S^c).
B29. [You do not need to know what *neighborhood* means] "If p $\in$ (0,1), then (0,1) is a neighborhood of p."
B31. "Let $f(x) = 2x$. f is increasing," is equivalent to "If $x < z$ and $f(x) = 2x$, then $f(x) < f(z)$."

B33. "If S is open, then S^c is closed."
B35. "If n is even, then n^2 is even."
B37. "$n + n^2$ is even"
B39. $2*5 = 12$. $x*3 = x + 6$. $6*x = 6 + 2x$.
$6 + 2x = 12(x + 6)$. $-66 = 10x$. $x = -6.6$.
B41. a) $(-\infty,2]$ b) (7, 10)
B43. If [(if $x \in$ R$\cap$T, then $x \in$ R) and (if $x \in$ R, then $x \in$ R$\cap$T)], then (if $x \in$ R, then $x \in$ T).
Proof: If $x \in R$, then $x \in R \cap T$ by hypothesis, so $x \in T$ by definition of intersection.

Chapter 5. Proofs

Section 5.0. Why Learn to do Proofs?

B1. Answered in text, but students often have other good ideas.
B3. [Your choice.] An example of how you can be misled.

Section 5.1. Proof

A1. Exhibit an x such that $H(x)$ is true and $C(x)$ is false.
A3. $x < z \Rightarrow 3x < 3z$ (T1.5.2) $\Rightarrow 3x + 8 < 3z + 8$ (T1.5.1)
A5. $|x - 2| < 5 \Rightarrow -5 < x - 2 < 5$ (T1.5.5) $\Rightarrow -3 < x$ (T1.5.1)
A7. $|2x - 1| < c \Rightarrow -c < 2x - 1 < c$ (T1.5.5) $\Rightarrow -c + 1 < 2x < c + 1$ (T1.5.1)
$\Rightarrow (-c + 1)/2 < x < (c+1)/2$ by T1.5.2
A9. F. Let $c = 2$ and $x = -1$.
A11. F. Let $x = -6$.
A13. T. $0 < 5 < x = |x|$.
A15. A is $(x - a)(x - b) = 0$. B is "$x - a = 0$ or $x - b = 0$," and C is "$x = a$ or $x = b$."
A17. $[(x \in R \Rightarrow x \in S)$ and $(x \in S \Rightarrow x \in T)] \Rightarrow (x \in R \Rightarrow x \in T)$.
The tautological form is "$[(A \Rightarrow B)$ and $(B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$."

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B1. a) A proof of a theorem is a sequence of statements which demonstrate that the theorem is a logical consequence of prior results. b) Tautological form and prior results.
B3. It must exhibit x such that $H(x)$ is true and $C(x)$ is false (and demonstrate that).
B5. F. Let $a = 4$ and $b = -6$.
B7. T. Its contrapositive is: $c > 0$ and $x > c \Rightarrow x^2 > c^2$. This was Example 4.
B9. a) False. Let $a = -3$ and $b = 1$. $a = -3 < 1 = b$, but $|a| = 3$ is not less than $|b| = 1$.
b) There exists a and b such that $a < b$ and $|a|$ is not less than $|b|$.
B11. a) False. Let $c = -4$ and $x = 2$. b) There exists c and x such that $|cx| \neq c|x|$.
B13. $x^2 = a^2$ iff $x^2 - a^2 = 0$ [Uniqueness of Addition]
iff $(x - a)(x + a) = 0$ [substitution]
iff $x - a = 0$ or $x + a = 0$ [ZPR] [or] $x = a$ or $x = -a$ by Example 5.1.1.
iff $x = a$ or $x = -a$ [UA, twice]
B15. $|x - 5| < 1 \Rightarrow -1 < x - 5 < 1$ (T1.5.5) $\Rightarrow x < 6$ (T1.5.1) $\Rightarrow 3x < 18$ (T1.5.2)
B17. $x < z \Rightarrow 5x < 5z$ (T1.5.2) $\Rightarrow 5x + 7 < 5z + 7$ (T1.5.1)
B19. $b < 0$ so $0 < 0 - b = -b$. Now use Part A. $(-b)c < (-b)d$. $-(bc) < -(bd)$. Add $bc + bd$ to get $bd < bc$.

^^^^^^^^

C1. An axiom system cannot be both consistent (no contradictions can be deduced) and complete (all statements can be proved or disproved). a) Maybe the Constitution is "inconsistent" and allows contradictions to be deduced by starting from different axioms within it. For example, maybe "freedom of speech" could lead to deductions that conflict with deductions from some other right in the Constitution, if both arguments are pressed far enough. Then the choice that prevails would depend upon which right is stressed in the arguments. The Justices can get to opposite conclusions by emphasizing different parts of the Constitution in their reasoning.
b) On the other hand, as an axiom system, the Constitution is "incomplete." There are issues

that can be raised that cannot be addressed by reference to the Constitution (e.g. Is blue or green a better color?).

Section 5.2. Proofs, Logic, and Absolute Values

A1. [p. 289] Theorems often use a letter to represent an infinite number of cases, and they often can be proved using that letter to represent all cases simultaneously in what is called a representative-case proof.

A3. A statement that is always true because of the arrangement of the connectives.

A5. a) $a = b$ b) $aa = ab$

A7. $-4 < x - 3 < 4 \Rightarrow -1 < x < 7$

A9. $-1 < 2x - 5 < 1 \Rightarrow 4 < 2x < 6 \Rightarrow 2 < x < 3$

A11. $|x - 7| < 1$

A13. $|x - 5| < 2$

A15. $|x + 4| < 1$

A17. $4 < x < 6$ iff $|x - 5| < 1$

A19. F, $x = -9$ A21. F, $x = 7$ A23. F, $x = -4$ A25. T

A27. F, $x = -30$ A29. F, $a = -3$, $b = -2$ A31. F, $a = 2$, $b = -3$

A33. F, $a = -2$, $b = -3$

A35. T A37. F, $c = -1$, $a = 1$ A39. F. $a = -3$, $b = 3$

A41. If x < z, then 4x < 4z by Theorem 1.5.2A. Then 4x+10 < 4z+10 by T1.5.1.

A43. Proof by exhaustion, checking 3 cases.

A45. (A $\Rightarrow$ B) iff (not B $\Rightarrow$ not A)

A47. "(A $\Rightarrow$ B) and (B $\Rightarrow$ C) and (C $\Rightarrow$ D) $\Rightarrow$ (A $\Rightarrow$ D)" is a tautology.

A49. $cx < 0$ because T1.5.2B. $|cx| = -(cx)$ [def of abs value of a negative number] $= c(-x)$ [arith] $= |c||x|$ def of abs value, twice.

A51. $(2x + 4)^2 = (x + 3)^2$. $4x^2 + 16x + 16 = x^2 + 6x + 9$. $3x^2 + 10x + 7 = 0$.
$(3x - 7)(x + 1) = 0$. $x = -1$ or $x = -7/3$, or use QF

A53. $|x| \leq c$ iff $-c \leq x \leq c$.

A55. Form and prior results.

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B1. $a < b \Rightarrow a + c < b + c$ (T1.5.1). $b + c < b + d$ (also T1.5.1). $a + c < b + d$ (transitive).

B3. "$a < b$ and $c < d \Rightarrow a - d < b - c$."
Proof: $c < d \Rightarrow -d < -c$. Now add and the result follows from T1.

B5. T6B. $x \geq 0$ or $x < 0$. If $x \geq 0$, $-x \leq 0$ [1.5.2B]
$|-x| = -(-x)$ [def of abs val] $= x = |x|$ [def of abs val]
If $x < 0$, $-x > 0$ [1.5.2B] $|-x| = -x$ [def of abs val] $= |x|$ [def of abs val]

B7. Use "Two conclusions."
$x \geq 0$ or $x < 0$. In the first case, $x \geq 0$,
$-|x| \leq 0$ [by T6A and T1.5.2B] $\leq x$ [given] $= |x|$ [def]. So, for $x > 0$, $-|x| \leq x \leq x$.
Now, in this, replace x by $-x$.
$-|-x| \leq -x \leq |-x|$. Using part B, $-|x| \leq -x \leq |x|$.
Multiplying by -1 (T1.5.2B), $|x| \geq x \geq -|x|$, as desired.

B9. False. $a = 1$, $b = 1$, $c = -2$.

B11. $-|x| \leq x \leq |x|$, by T6. Ditto for y. Adding
$-(|x| + |y|) \leq x + y \leq |x| + |y|$ by the "$\leq$" variant of Theorem 1.
Now $|x + y| \leq |x| + |y|$ by T8, the Theorem on Absolute Values.

B13. False. $x = 1$, $a = 2$ $y = 2$, $b = 3$. Then $a - b$ is negative.

B15. False. $x = 6$ and $y = 8$.

B17. True. $y + (x - y) = x$. $|y| + |x - y| \ge |y + (x - y)| = |x|$ by T11, the triangle inequality.
Subtracting $|y|$ yields $|x - y| \ge |x| - |y|$.

B19. Case $a < b$. If $a < b$, $a - b < 0$ and $|a - b| = b - a$.
$(1/2)(a + b + |a - b|) = (1/2)(a + b + b - a) = (1/2)(2b) = b$, the larger of a and b.

B21. A fix: $c \ge 0$ implies ($|x| = c$ iff $x = c$ or $x = -c$).
Proof of "$\Rightarrow$": $x \ge 0$ or $x < 0$. If $x \ge 0$, $|x| = c$ implies $x = c$. $x < 0 \Rightarrow c = |x|$ [given] $= -x$ [def of abs value] $\Rightarrow x = -c$ [UM, mult by -1]
Proof of "$\Leftarrow$": $x \ge 0$ or $x < 0$. Case I: $x = c$ [given] ≥ 0 [given] $\Rightarrow |x| = x$ [def] $= c$.
Case II: $x = -c$ [given] ≤ 0 [T1.5.2B var] so $|x| = -x = c$ and $x = -c$.

[or] The fix could be: $|x| = c$ iff $c \ge 0$ and $x = c$ or $x = -c$.
Proof of "$\Rightarrow$": $c \ge 0$ by T6A. : $x \ge 0$ or $x < 0$. If $x \ge 0$, $|x| = c$ implies $x = c$. $x < 0 \Rightarrow c = |x|$ [given] $= -x$ [def of abs value] $\Rightarrow x = -c$ [UM, mult by -1]
Proof of "$\Leftarrow$": $x \ge 0$ or $x < 0$. If $x \ge 0$, $|x| = x = c$. If $x < 0$, $|x| = -x$ and $x = -c \Rightarrow x = -c$ $\Rightarrow |x| = c$.

B23. True. Proof of "$\Leftarrow$": $x^2 = y^2$ iff $x^2 - y^2 = 0$ iff $(x - y)(x + y) = 0$ iff $x = y$ or $x = -y$. In either case $|x| = |y|$, by T5B.
Proof of "$\Rightarrow$": $|y| \ge 0$. $|x| = |y| \Rightarrow x = |y|$ or $x = -|y|$ by C21 $\Rightarrow x = \pm y$ or $x = -(\pm y)$.
In either case, $x^2 = y^2$. [$x = y \Rightarrow x^2 = y^2$. $x = -y \Rightarrow x^2 = (-y)^2 = y^2$.]

B25. $|x + y| \ge ||x| - |y||$ is true. Proof: $|x| = |x + y - y| \le |x + y| + |-y| = |x + y| + |y|$. Subtracting $|y|$, $|x| - |y| \le |x + y|$. The same fact holds with the letters switched: $|y| - |x| \le |y + x|$.
But the left side is $-(|x| - |y|) \le |x + y|$. The right side is greater than or equal to a number and its negative, so it is greater than or equal to the absolute value, as desired.

B27. The ones that do are: T1, Conj 3, T8, T9, Conj 24.

B29. True. $a < b \Rightarrow -a > -b$ [T1.5.2B] $\Rightarrow c + (-a) > c + (-b)$ [T1.5.1]. $c - a > c - b$.

B31. $a < b \Rightarrow a + b < b + b$ [T1.5.1] $\Rightarrow (a+b)/2 < b$ [T1.5.2A]

B33. No. It is still false. Half is true. Let $x = 0$, $c = 5$ and $d = -2$. Then $|x| \le d + c$, but $|x - c|$ is not $\le d$.
[Unnecessary] Proof of "$\Rightarrow$": $|x - c| < d \Rightarrow c - d < x < c + d$ [T9]
$-(c + d) = -c - d < c - d$ [because $c > 0$] $< x$ [from before], so $|x| < c + d$ by T8.

B35. True. Begin with T7. Negate both parts.
b) Yes. All x's satisfy both sides.

B37. $x^2 = a^2$ iff $x^2 - a^2 = 0$ [Uniqueness of Addition]
iff $(x - a)(x + a) = 0$ [substitution]
iff $x - a = 0$ or $x + a = 0$ [ZPR] [or] $x = a$ or $x = -a$ by Example 5.1.1.
iff $x = a$ or $x = -a$ [UA, twice]
[abbreviated] $a^2 = b^2$ iff $a^2 - b^2 = 0$ iff $(a-b)(a+b) = 0$ iff $a-b = 0$ or $a+b = 0$ iff $a = b$ or $a = -b$.

B39. False. $a = 3$, $b = -5$, $c = 5$.

B41. $|f(x) - 13| = |5x - 7 - 13|$ [hyp] $= |5x - 20| = 5|x - 4|$ [T6C] $< 5(1) = 5$ [hyp and 1.5.2A]

B43. $|x - p| < k$ is $p - k < x < p + k$.

B45. The lesser of a and b is $(1/2)(a + b - |a - b|)$.
Case $a < b \Rightarrow a - b < 0 \Rightarrow |a - b| = b - a$ and $(1/2)(a + b - |a - b|)$
$= (1/2)(a + b - (b - a)) = a$.

B47. Y	B49. Y	B51. Y	B53. N
B55. Y	B57. N	B59. N	B61. Y

B63. Without loss of generality call the greater one x, so $x \ge y$.
$x + y \le x + x = 2x$. $y > 2 \Rightarrow xy > 2x > x + y$.

^^^^^^^^

C1. Let $c > 0$. Chose $d = c/4$.
Then $|f(x) - 10| = |4x - 2 - 10| = |4\mathrm{x} - 12| = 4|x - 3| < 4d = 4(c/4) = c$.

C3. $|2x - 1| = 7$, $2x - 1 = 7$ or $2x - 1 = -7$, $2x = 8$ or $2x = -6$, $x = 4$ or $x = -3$.
$|x - 2| = -3$ ha no solutions.
Theorem: $|x| = c$ iff $c \geq 0$ and ($x = c$ or $x = -c$).

Section 5.3. Translation and Organization

A1. a) If $x \in S$, then $x \in T$. b) If $x < z$, then $g(x) \leq g(z)$. c) If $x \in S$, then $x \leq 45$.
A3. a) $x \in S$ or $x \in T$. b) $x \in S$ and $x \in T$.
A5. $x \in S \cap T$ A7. $x < z$
A9. (C and A and B) $\Rightarrow D$. A11. [C and ($A \Rightarrow B$)] $\Rightarrow D$.
A13. a) It is tautological.
b) If x in S, then x in S or T is equivalent to If x in S, then x in $S \cup T$,
which is equivalent to $S \subset S \cup T$

^^^^^^^^

B1. Terms are used in a context, and sentences provide that context. Also, logic applies to sentences (not terms) so we can exhibit logical connectives by replacing an entire sentence with a term with an equivalent new sentence exhibiting logical connectives.
B3. "Or" in the Conclusion. 3.3.5
B5. "Iff" Conclusions. 3.3.11
B7. Cases. 3.2.9
B9. no such
B11. Two Conclusions. 3.2.6
B13. A Hypothesis in the Conclusion, 3.2.7.
B15. A Hypothesis in the Conclusion, 3.2.7.
B17. A Hypothesis in the Conclusion, 3.2.7.
B19. "Iff" Conclusions. 3.3.11
B21. $x \in S \Rightarrow x \leq 20$.
B23. For all x, $x \in S^c \cup T$.
B25. $R \cap S = S$ translates to $x \in R \cap S$ implies $x \in S$ [easy] and $x \in S$ implies $x \in R \cap S$.
B27. If ($x \in S$ or $x \in$ R), then ($x \in T$ or $x \in$ R)
B29. $0 \leq x < z \Rightarrow f(x) < f(z)$
B31. $x \in S \Rightarrow |x| \leq 20 \Rightarrow x \leq 20$ [T1.5.5]
B33. Possibly start with "Let $x \in U$." [Why start there?] For all x, $x \in S$ or $x \in S^c$.
$x \in S \Rightarrow x \in T$. Therefore, $x \in S^c$ or $x \in T$, i.e. $x \in S^c \cup T$.
[or] Then $x \in S$ or $x \in S^c$. If $x \in S$, then $x \in T$ and $x \in S^c \cup T$. If $x \in S^c$, then $x \in S^c \cup T$.
B35. To show if $x \in S$, then $x \in R \cap S$ [because the other half is trivial].
$x \in S \Rightarrow x \in R$ [hyp] so $x \in R \cap S$.
B37. Let $x \in S$ or $x \in R$. If $x \in S$, then $x \in T$, so $x \in T$ or $x \in R$.
B39. Let $0 \leq x < z$. Then $x^2 < z^2$ by Example 5.1.4, so $f(x) < f(z)$.
B41. $R \Rightarrow R \cup S$.
B43. a) If (if $x \in$ R, then $x \in$ S) and (if $x \in S$, then $x \in T$), then (if $x \in$ R, then $x \in T$).
b) (A $\Rightarrow B$) and ($B \Rightarrow$ C) $\Rightarrow$ (A $\Rightarrow$ C) c) Yes, it is transitivity of $\Rightarrow$.
B45. Translation: not($x \in S$ or $x \in T$) iff ($x \notin S$ and $x \notin T$).
This is simply an instance of DeMorgan's laws and definitions of set-theory terms.
B47. $x < z \Rightarrow 3x < 3z \Rightarrow f(3x) < f(3z) \Rightarrow 2f(3x) < 2f(3z) \Rightarrow g(x) < g(z)$).
B49. Use A Hypothesis in the Conclusion." Let $x < z$. Then $f(x) < f(z)$ and $f(x) + c < f(z) + c$, so $g(x) < g(z)$.
B51. Let $x \in S$. Then $x \in T$ and $x \leq 17$.

B53. a) If there exists (a, b) such that $p \in (a, b) \subset N$ and there exists (c, d) [don't use (a, b) again] such that $p \in (c, d) \subset M$, then there exists (r, s) such that $p \in (r, s) \subset M \cap N$.
b) Let $r = \max\{a, c\}$ and $s = \min\{b, d\}$. Then $p \in (r, s) \subset M \cap N$. [You could add more details.]

B55. a) $(A \Rightarrow B)$ iff (not $B \Rightarrow$ not A). b) $R \subset S$ iff $S^c \subset R^c$.

B57. If $x \in S \Rightarrow x \in T$, then ($x \in S$ and $x \in R$) $\Rightarrow$ ($x \in T$ and $x \in R$).
[or] Let $x \in S \cap R$). The $x \in S$ and $x \in R$. By hyp, $x \in T$. So $x \in T$ and $x \in R$.

B59. a) $A \Rightarrow$ (B iff C). b) $b > 0$ and $c < d \Rightarrow bc < bd$ and $b > 0$ and $bc < bd \Rightarrow c < d$.
c) the first half is 1.5.3.
d) [Claim] $b > 0 \Rightarrow 1/b > 0$. $bc < bd \Rightarrow (1/b)bc < (1/b)bd$, $c < d$.

^^^^^^^^

C1. a) Let $R \in P(S)$. Then $R \subset S$, by def. $S \subset T$, by hyp. Thus $R \subset T$, so $R \in P(T)$.
b) $S \in P(S) \subset P(T)$, so $S \in P(T)$, so $S \subset T$.
[alternative] b) Let $x \in S$. Then $\{x\} \subset P(S) \subset P(T)$, so $\{x\} \subset P(T)$ and $x \in T$.

C3. a) c is the least element in S iff (1) $c \in S$ and, if $x \in S$, then $x \geq c$.
b) No. For all $c \in S$, there exists $x \in S$ such that $x < c$ (e.g. $c/2$).
c) Yes. 1. d) Yes. 1.
f) Proof: If there does not exist such an element, then $x \in S \Rightarrow x \leq c$, and c would be an upper bound less than 10, so 10 would not be the least upper bound. [Proof by contrapositive (or contradiction)]

C5. Let M be $\sup(S \cup T)$. If $x \in S$, then $x \in S \cup T$ and $x \leq M$, so M is an upper bound of S and the least upper bound of S must be $\leq M$.

C7. The hyp shows b is an upper bound. The least upper bound must be $\leq b$.

Section 5.4. The Theory of Proofs

A1. A proof is indirect if it employs as a hypothesis the negation of the conclusion or the negation of a component of the conclusion.

A3. Because they automatically have the same truth value.

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B1. Roughly, a proof of a statement is a sequence of statements which demonstrate that the statement is a logical consequence of prior results.

B3. To "deduce" means to use logic properly to draw a conclusion from given hypotheses (which may be false). To "prove" means to use logic properly to draw a conclusion from prior results. For proof, even using "true" results for steps is not acceptable if they are not prior.

B5. A proof does **not** need to begin with a hypothesis. The steps consist of true statements from the list or hypotheses, but which is first depends upon the organization of the proof which is determined more by the natural logical organization (the "flow" of the steps) than it is by which steps are hypotheses and which are not.

B7. Instead of using "$P \Rightarrow (H \Rightarrow C)$" we use "($P$ and H) $\Rightarrow C$ ", which is logically equivalent. Then P is like a prior result for proving C, but we are not really proving C, we are proving $(H \Rightarrow C)$

B9. a) Not necessarily. (Without variables, the meaning is quite different. In a truth table, "2+2 = 4 $\Rightarrow$ the Fundamental Theorem of Calculus" would come up "True," but not even in regular math.) b) Yes, that is the Math usage. c) No.

B11. Yes. B13. No. B15. No. B17. Yes.

B19. True. $A \wedge B \Rightarrow A \Rightarrow C$.
B21. True.
B23. True, by Cases.

B25. No. B27. Yes. B29. No. B31. Yes. B33. Yes. B35. No.

B37. Short way: $A \Rightarrow A$ or B [taut] $\Rightarrow C$ [hyp]
[Done, by hypotheses in the conclusion]
A longer way: D and E $\Rightarrow$ E (tautology). Now sub $A \Rightarrow C$ for D and $B \Rightarrow C$ for E to get $(A \Rightarrow C)$ and $(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$. Now rewrite the left side using "cases"
$[(A$ or $B) \Rightarrow C] \Rightarrow (A \Rightarrow C)$.
B39. A and $B \Rightarrow B$ [taut] $\Rightarrow C$. [hyp]
B41. *A* and not(*B* and *C*) LE *A* and ((not *B*) or (not *C*))
LE (*A* and (not *B*)) or (*A* and (not *C*)). b) Yes.
B43. All "no" except C1.
B45. C1, 3, and 6 follow.
B47. C2, C5, C6 follow.
B49. none
B51. nothing very firm, but we can deduce: If it is non-Abelian it is of even order [or] It is either Abelian or of even order.
B53. if it is simple it is of even order [or] it is not (simple (and) of odd order).
B55. nothing
B57. it is of even order
B59. it is not non-Abelian (it is Abelian)
B61. nothing

B63. Only C8 follows.
B65. Yes. $A \wedge B \Rightarrow B \Rightarrow C$. $A \wedge C \Rightarrow D$. Thus $C \wedge D$. $C \wedge D \Rightarrow R$ [from the "or" hypothesis].
B67. Yes. $A \Rightarrow B$ or C. If $A \Rightarrow B$, we are done. If $A \Rightarrow C$, then $C \Rightarrow D$.
B69. No. A is T, *B* is F, D is F, and C is F.

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C1. "How do proofs work?" [essay not given here]
C3. $x^2 + 2dx + k = 0$ iff $x^2 + 2dx + d^2 - d^2 + k = 0$
iff $(x + d)^2 - d^2 + k = 0$ [identity, substitution]
iff $(x + d)^2 = d^2 - k$ [UA]
iff $x + d = \pm\sqrt{(d^2 - k)}$ [Rule 2.3.6]
iff $x = -d \pm\sqrt{(d^2 - k)}$. [UA]
b) Since $a \neq 0$, $ax^2 + bx + c = 0$ iff $x^2 + (b/a)x + (c/a) = 0$. So replace d with $b/(2a)$, and k with c/a, and simplify.
C5. The "proof" starts with the conclusion. That is unacceptable. The "proof" proves the converse.
C7. Proofs have two components, logic and prior results. Your steps must be prior *on the current list*. So the student will have to inform the prof what is on his or her list. Maybe the cleverest proof uses a step that they student can't use because it is not on the list. Or maybe the professors proof uses elementary steps when there was a shorter and more elegant proof from a result the prof the student had on his or her list that the prof didn't know the student could use.

Section 5.5. Existence Statements and Existence Proofs

A1. False. Let a = -10.
A3. False. $a = -10 < 2 = b$, but $|-10| > |2|$.
A5. True. $a > 0$ and $a < b \Rightarrow b > 0$ so $|a| = a$ and $|b| = b$. Therefore $|a| < |b|$.

^^^^^^^^
B1. Finding a candidate x, and then showing it really does satisfy $S(x)$.
B3. 55 is a bound. $|x\text{-}50| < 5$ iff $-5 < x - 50 < 5$ iff
$45 < x < 55 \Rightarrow |x| < 55$.
B5. Let b be the bound of S. Then $5b$ is the bound of T. Proof: $x \in T \Rightarrow x = 5s$ for some $s \in S$.
$|x| = |5s| = 5|s| \leq 5b$.
B7. Let $d = 1/3$. Then, $f(x) - f(y) = 3x - 3y = 3(x - y) < 3(1/3) = 1$, if $x - y < 1/3$.
B9. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/5$. Then $|f(x) - 7| = |5x - 3 - 7| = |5x - 10|$
$= 5|x - 2| < 5\delta = 5(\varepsilon/5) = \varepsilon$.
B11. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/5$. Then $|5x - 2 - 18| = 5|x - 4| < 5\delta = 5(\varepsilon/5) = \varepsilon$.
B13. Two case: $c = 0$ and $c \neq 0$. Let $\varepsilon > 0$. If $c = 0$, any δ will do because the left side is 0.
Let $c \neq 0$. By hyp, there exists $\delta > 0$ such that $|f(x) - L| < = \varepsilon/|c|$ if $|x - a| < \delta$.
Then $|cf(x) - cL| = |c||f(x) - L| < |c|\delta = |c|\varepsilon/|c| = \varepsilon$.
B15. By hyp and def 2, there exist m and j such that $n = 2m$ and $k = 2j$. Then $n+k = 2m + 2j = 2(m + j)$, which is even by the def of *even*.
B17. By hyp and def 2, there exists m such that $n = 2m$. Then $nj = (2m)j = 2(jm)$ is even by def 2.
B19. Contrapositive of B15.
B21. False. $a = 0$, $b = -3$, $c = 2$.
B23. $n^3 - m^3 = (n - m)(n^2 + nm + m^2)$. The first factor is ≥ 2 by hyp. So is the second.
Since it is a product or factors > 1, it is not prime.
B25. $z = ky$, where $k \neq 0$, by hyp and def of "proportional". Also $y = cx$, where $c \neq 0$, by hyp.
Thus $z = ky = k(cx) = (kc)x$, where $kc \neq 0$. [The "existence" nature has been suppressed, but is really there, and important.]
B27. True. $x = i/j$, where $j \neq 0$. $y = m/n$, where $n \neq 0$. Then $xy = (im)/(jn)$, where $jn \neq 0$.
B29. False. Let $x = 0$ and $y = \sqrt{2}$.
B31. True. By contrapositive: If $\sqrt{x}$ is rational it is p/q so $x = (p/q)^2 = p^2/q^2$.
B33. Let $r = (x + y)/2$. 5.2 B30 and B31 show it is between them.
B35. True. Let b be a bound of S and c be a bound of T. Let the greater of b and c be d, which is a bound of $S \cup T$. $x \in S \cup T$ implies $x \in S$ or $x \in T$ implies $x \leq b \leq d$ or $x \leq c \leq d$.

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C1. Let $\varepsilon > 0$. Choose δ to be the smaller of 1 and $\varepsilon/9$. Then
$|f(x) - 16| = |x^2 - 16| = |x - 4||x + 4| < 9|x - 4|$
since $|x + 4| < 9$ when $|x - 4| < 1$, which it is, by hyp $< 9\delta \leq 9(\varepsilon/9) = \varepsilon$.
C3. Let $\varepsilon > 0$. There exists δ_f such that $|x - a| < \delta_f \Rightarrow |f(x) - L| < \varepsilon/2$.
Similarly, there exists δ_g such that $|x - a| < \delta_g \Rightarrow |g(x) - M| < \varepsilon/2$.
Let δ = minimum of δ_f and δ_g. If $|x - a| < \delta$, then both conditions above hold.
$|(f(x)\pm g(x)) - (L\pm M)| = |(f(x) - L) \pm (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
C5. $f(x) = mx + b$. $f(f(x)) = m(mx + b) + b = m^2x + mb + b$. a line with slope m^2.

^^^^ C6-12. These just give the candidate. The rest of the proof is omitted.

C7. Let $f(x) = x + ½$.	C9. Let $f(x) = x^3$
C11. Let $f(x) = \sqrt{x}$	C13. Let $f(x) = 2^x$.

Section 5.6. Proof by Contradiction or Contrapositive

A1. Form: "A and B $\Rightarrow$ C," where A is "n pigeons are in k holes," B is "$k < n$," and C is "At least one hole has at least two pigeons." Proof form: (not C) and A $\Rightarrow$ (not B). This is really proof by a version of the contrapositive.

A3. No. If it has 9, it has 2. We only need at least two.

A5. There exists $x < z$ such that $f(x) \geq f(z)$.

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B1. [p. 329, introduction.] "Contradictions are necessarily false. If, by supposing "not A" a contradiction can be deduced, then "not A" must be false and "A" must be true.

B3. False. Let $x = 2 = c$.

B5. If both a and b are odd, then a^2 and $b^2 = (2k + 1)^2 + (2j + 1)^2$ for some i and j, $= 4(k^2 + k + j^2 + j) + 2$. This is not possible by B4.

B7. If $x + y \leq 1$, then $x < 1$ and $y < 1$ and $x^2 < x$ and $y^2 < y$, so $x^2 + y^2 < x + y = 1$.

B9. [Assume unique prime factorization as a prior result.] If $\sqrt{3}$ is rational, let it be i/j in lowest terms. Then $3 = (i/j)^2$. Then $3j^2 = i^2$, so i^2 is divisible by 3. So i must have factor of 3 [3 is a prime]. $i = 3k$ for some k. $3j^2 = (3k)^{2..}$ j^2 is divisible by 3. So j must have factor of 3. So both I and j have a factor of 3, contradicting our assumption of lowest terms. So $\sqrt{3}$ is not rational.

B11. T is unbounded means, for all b, there exists $x \in T$ such that $|x| > b$.
a) For any b, there exists $x \in S$ such that $|x| > b$, since S is unbounded.
Since $x \in S$ implies $x \in T$, it serves as the member of T such that $|x| > b$.
b) Suppose T is bounded. Then, there exists b such that, for all $x \in T$, $|x| \leq b$. Since $S \subset T$, for all $x \in S$, $|x| \leq b$, so S is bounded.
c) [Part (b) already did a version of the contrapositive. If we add in one more line saying that contradicts the hypothesis that S is not bounded, we could call it "by contradiction"]
d) The direct proof addresses the original organization. The first two are of about equal difficulty. Part (b) might be called "proof by contradiction," but there is no need to think of it that way. It is really proof by contrapositive.

B13. If 4 were an interior point then there would exist an interval (a, b) such that $4 \in (a, b) \subset \{3.9, 4, 5.1\}$. Then the interval would contain a point, for example, either 3.95 (if $a < 3.95$) or $(a+4)/2$ (if $a \geq 3.95$) that is not in the set.

B15. False. Claim: Any open interval about 3 has irrational numbers in it.

B17. The conclusion is "at least one hole has at least i+1 pigeons."
Its negation is "All holes have at most i pigeons." In which case here are at most ik pigeons.
[This negates the hypothesis, so this is a proof by contrapositive.]

B19. Conclude "$x \leq 2$". Suppose $x > 2$. Let $c = x - 2$. Then $x = 2 + c$, so not$(x < 2 + c)$.

B21. Contrapositive of B8. If c is odd, $c = 2k + 1$ and $c^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$ is odd.

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C1. Consider "[A and (not A)] $\Rightarrow$ [$H \Rightarrow C$]." Since the hypothesis is automatically false, this is a tautology. Given its hypothesis, the conclusion would follow [but it cannot be proper to be given the hypothesis.]

C3. We know $\sin^2 x + \cos^2 x = 1$. The result follows from B7.

C5. a) If x is odd, x^2 is odd and the total would be odd, not 0. So, x is not odd.
b) If x is even, $x^2 = 2m(2k) + 2(2j + 1) = 4b + 2$, so some b. By B4, x cannot be an integer.
c) Let x be p/q in lowest terms. $(p/q)^2 + 2m(p/q) + 2n = 0$. $p^2 + 2mpq + 2nq^2 = 0$.
If q is odd, this has the same form as part (a), with p being the x.
If q is even, p is odd, p^2 is odd the total cannot be 0, and even number.

Section 5.7. Mathematical Induction.

A1. $[n(n+1)/2] + (n+1) = (n+1)[(n/2) + 1]$
A3. $1.02^3 \geq 1 + 3(.02) = 1.06$.

^^^^^^^^

B1. We are assuming $S(n)$ to help prove T2(2), not directly to prove T2(3), which is a deduction from the theorem.

B3. $(1 + a)^3 = 1 + 3a + 3a^2 + a^3 > 1 + 3a^2$ if $a > 0$.
$(1 + a)^{n+1} = (1 + a)(1 + a)^n > (1 + a)(1 + na^2)$ by the induction hypothesis
$= 1 + a + na^2 + na^3$. If $0 < a \leq 1$, $a \geq a^2$ and the first three terms $\geq 1 + (n+1)a^2$.
If $a > 1$, $a^3 > a^2$ and the final two terms $\geq (n + 1)a^2$.

B5. Check $n = 1$. $(1 - r^2)/(1 - r)$ does $= 1 + r$, for $r \neq 1$.
Let $S(n)$ be as given. Add r^{n+1} to both sides. The left side will then be as desired. The right side will be $(1 - r^{n+1})/(1 - r) + r^{n+1}$. Using a common denominator,
$= (1 - r^{n+1})/(1 - r) + (1 - r)r^{n+1}/(1 - r) = (1 - r^{n+2})/(1 - r)$, as desired.

B7. Let $S(n)$ be $f(n) < 2$. For case 1: $f(1) < 2$ is given.
$f(n+1) = f(n)/2 + 1 < 2/2 + 1$ [by induction hypothesis] $= 2$.

B9. Let $S(n)$ be "$x_n < 2$." $S(0)$ is true by inspection.
a) $x_{n+1} = \sqrt{(2 + x_n)} < \sqrt{(2 + 2)}$, since $x_n < 2$ (by the induction hypothesis and the fact that $\sqrt{}$ is an increasing function) $= 2$.
b) $x_{n+1} = \sqrt{(2 + x_n)} > \sqrt{(x_n + x_n)}$ [since $x_n < 2$ by part (a) and $\sqrt{}$ is an increasing function]
$= \sqrt{(2x_n)} > \sqrt{(x_n(x_n))} = x_n$ (since $x_n \geq 0$ since it is a square root).
[Note: (b) follows from this and (a) without another use of induction]

B11. Let $S(n)$ be "$x_n > 4$." $S(0)$ is true by inspection.
a) $x_{n+1} = \sqrt{(12 + x_n)} > \sqrt{(12 + 4)}$ [since $x_n > 4$ and $\sqrt{}$ is increasing] $= 4$.
b) [Part (b) uses a second induction proof.] Let $S(n)$ be " $x_{n+1} < x_n$."
$x_{n+2} = \sqrt{(12 + x_{n+1})} < \sqrt{(12 + x_n)}$ [by the induction hypthesis and $\sqrt{}$ is increasing] $= x_{n+1}$

B13. Let $S(n)$ be $4^n - 1$ is divisible by 3. For $n = 1$, $4^1 - 1 = 3$ which is divisible by 3.
$4^{n+1} - 1 = 4(4^n) - 1 = 3(4^n) + (4^n - 1)$, both terms of which are divisible by 3, by the induction hyp.

B15. True. Almost the same proof. For the base case, 5|5.
$6^{n+1} - 1 = 6(6^n) - 1 = 5(6^n) + (6^n - 1)$, both terms of which are divisible by 5, by the induction hyp.

B17. For $n = 1$, $n^3 + 5n + 6$ is 12, which is divisible by 3.
$(n+1)^3 + 5(n+1) + 6 = n^3 + 3n^2 + 3n + 1 + 5n + 5 + 6 = (n^3 + 5n + 6) + 3(n^2 + n + 2)$, both terms of which are divisible by 3, by the induction hypothesis.

B19. For $n = 1$, 9|9. $10^{n+1} - 1 = 10(10^n) - 1 = 9(10^n) + (10^n - 1)$, both terms of which are divisible by 9, by the induction hypothesis.

B21. Prove: If $a+b$ is divisible by 3, then $a(10^n)+b$ is divisible by 3.
For $n = 0$, it is trivial. $a(10^{n+1}) + b = 10a(10^n) + b = 9a(10^n) + [a(10^n) + b]$, a sum of terms divisible by 3, by the induction hypothesis.

B23. Check $n = 1$. $1/5 = 1/5$ is true. Let $S(n)$ be as given. Add the next term to both sides.
The right side becomes $n/(4n + 1) + 1/[(4n + 1)(4n + 5)]$.
Use the common denominator:
$n/(4n + 1) + 1/[(4n + 1)(4n + 5)]$
$= n(4n + 5)/[(4n + 1)(4n + 5)] + 1/[(4n + 1)(4n + 5)]$
$= (4n^2 + 5n + 1)/[\ ] = (4n + 1)(n + 1)/[\ ] = (n+1)/(4n+5) = (n + 1)/(4(n+1)+1)$.

B25. Let $S(n)$ be the given statement. For case 1: $1(1!) = 1$. $(1+1)! - 1 = 2 - 1 = 1$.
$1(1!) + 2(2!) + \ldots n(n!) + (n+1)(n+1)! = [(n+1)! - 1] + (n+1)(n+1)!$
[by the induction hypothesis]

$= (n+1)![1 + (n+1)] - 1 = (n+2)! - 1$, as desired.

B27. Cse 1 and 2 work by inspection. Assume case n and lower hold. Then
$a_{n+1} = 3a_n - 2a_{n-1}$ by hypothesis $= 3(2^n - 1) - 2(2^{n-1} - 1)$ by the IH $= 2^{n+1} - 1$ [by algebra]

B29. For the base case, $1^3 = (1(1+1)/2)^2 = 1$.
$1^3 + 2^3 + ... + n^3 + (n+1)^3 = (n(n+1)/2)^2 + (n+1)^3$ by IH
$= (n+1)^2 [n^2/4 + (n+1)]$
$= (n+1)^2 [n^2 + 4n + 4]/4 = (n+1)^2(n+2)^2/4 = ((n+1)(n+1+1)/2)^2$

B31. Yes. $(1 + a_1) ... (1 + a_n) \geq 1 + a_1 + ... + a_n$ if $-1 < a_i \leq 0$ for all i.

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C1. The integers. (This set doesn't have a least element).
[or] Consider $\{1, 2, 3, ...\} \cup \{a\}$, where the successor of a is 2, but $a \neq 1$.
Then numbers don't have unique predecessors.

C3. Assume $u_n \geq 0$ for all n. Assume the base case holds.
Then $u_{n+1} \leq (1 - (2/n) + 1/n^2)u_n \leq (1 - (2/n) + 1/n^2)(1/(n-1)^2)$
$= [(n^2 - 2n + 1)/(n^2)](1/(n-1)^2) = 1/n^2$.

C5. Prove: $10^n + 3(4^{n+2}) + 5$ is divisible by 9.
For $n = 1$, it is 207 which is 9(23).
$10^{n+1} + 3(4^{n+3}) + 5 = (9+1)10^n + (3+1)3(4^{n+2}) + 5$
$= 9(10^n) + [10^n + 3(4^{n+2}) + 5] + 3(3)(4^{n+2})$, a sum of terms divisible by 9.

C7. $8 = 3 + 5$. $9 = 3+3+3$, and $10 = 5 + 5$.
Consider $n \geq 10$. Let it be true for n and all integers between 8 and n.
Then $n+1 = n-2 + 3$, and $n-2 \geq 8$ and can be written as a sum of 3's and 5's, so
$n+1$ can be written as a sum of 3's and 5's.

C9. Case 1 is stated wrongly. The point is not that n is odd, but n^2+n. For $n = 1$, that is even, so the proof (and result) is wrong.

C11. This is nearly trivial. All that happens is a renumbering. Suppose you want to prove "$S(n)$, for all $n \geq k$," where $k > 1$. Let $T(n) = S(n-k+1)$. Then $S(k)$ is $T(1)$ and "$n \geq k$" for $S(n)$ is the same as $n \geq 1$ for $T(n)$. So, the given statement of induction, which proves $T(n)$, $n \geq 1$, with base case 1, also proves $S(n)$, $n \geq k$, with base case k.